THE RESTRICTION THEOREM FOR FULLY NONLINEAR SUBEQUATIONS

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ABSTRACT

Let X be a submanifold of a manifold Z. We address the question: When do viscosity subsolutions of a fully nonlinear PDE on Z, restrict to be viscosity subsolutions of the restricted subequation on X? It is not always true. In this paper we formulate a general result which guarantees good restriction. This is then applied to a long list of analytically and geometrically interesting cases.

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1. Introduction

This paper is concerned with the restrictions of subsolutions of a fully nonlinear elliptic partial differential equation to submanifolds. In most cases this topic is uninteresting because the restricted functions satisfy no constraints. Moreover, even when there are constraints, this will occur only on certain submanifolds. Nonetheless, there are cases, in fact many cases, where the restriction question is quite interesting. Important classical examples are the plurisubharmonic functions in several complex variable theory, and their analogues in calibrated geometry. The principle aim of this paper is to study the foundations of the restriction problem. We prove a general Restriction Theorem, and then make a series of applications. Some come from potential theory developed in calibrated and other geometries (cf. [$\mathrm{HL}_{2,3}$]). Others come from universal subequations in riemannian geometry (cf. [$\mathrm{HL}_{6,7}$]). Yet another application will be to the study of the intrinsic potential theory on almost complex manifolds (without use of a hermitian metric).

We begin with a note about our approach to this problem. Traditionally, a second-order partial differential equation (or subequation) is a constraint on the full second derivative (or 2-jet) of a function u imposed by using a function $\mathbf{F}(x, u, Du, D^2u)$ and setting $\mathbf{F} = 0$ (or $\mathbf{F} \geq 0$). We have found it more enlightening to work directly with the subsets of the 2-jet space corresponding to these conditions (cf.[K]), and we have systematically explored this viewpoint in recent papers [$\mathrm{HL}_{4,5,6}$]. This geometric formulation is often more natural and has several distinct advantages. To begin, it makes the equation completely canonical. It clarifies a number of classical conditions, such as the condition of degenerate ellipticity. It underlines an inherent duality in the subject, which in turn clarifies the necessary boundary geometry for solving the Dirichlet problem.

It also simplifies and clarifies certain natural operations, in particular those of restriction and addition.

To be more concrete, let's begin with a closed subset F of the space of 2-jets over a domain $Z \subset \mathbf{R}^n$, which we assume to satisfy the weak positivity condition (2.4) below. Such a set will be called a *subequation*. Then a function $\varphi \in C^2(Z)$ is called *F-subharmonic* if its 2-jet $J^2_x\varphi \in F$ for all x. This concept can be extended to upper semi-continuous functions u by using the following viscosity approach (cf.[CIL], [C]). We say that a function φ which is C^2 near $x \in Z$ is a *test function* for u at x if $u \leq \varphi$ near x and $u(x) = \varphi(x)$. Then a function $u \in \mathrm{USC}(Z)$ is *F-subharmonic* if for each test function φ for u at any $x \in Z$, one has $J^2_x\varphi \in F$.

Suppose now that

$$F \subset J^2(Z)$$

is a subequation and $i : X \subset Z$ is a submanifold of Z. Then there is a naturally induced subequation

$$i^*F \subset J^2(X)$$

given by the restriction of 2-jets (which is induced by the restriction of functions). By definition it has the property that for $\varphi \in C^2(Z)$

$$\varphi \ \text{ is } F\text{−subharmonic} \qquad \Rightarrow \qquad \varphi\big|_X \ \text{ is } i^*F\text{−subharmonic} \tag{1.1}$$

For generic F and X the induced subequation will be vacuous, i.e., $i^*F = J^2(X)$. This leads to two natural problems.

Problem 1: Identify non-vacuous cases and calculate the induced subequation i^*F.

Once accomplished, we have the second, more difficult:

Problem 2: Find conditions under which the restriction statement (1.1) extends to upper semi-continuous functions.

In the classical case coming from several complex variable theory, the subequation F is defined by requiring that the complex hermitian part of the hessian matrix be non-negative. Here the most interesting submanifolds are the complex curves, and in this case the restricted subequation is the conformal Laplacian. Thus the prototype of our main result is the theorem which says that a function which is plurisubharmonic in the viscosity sense is the same as a function whose restriction to every complex curve is subharmonic. An even more basic case is the real analogue, which states that a function is convex in the viscosity sense if and only if its restriction to each affine line is convex. These classical cases are related to the study of the homogeneous Monge-Ampère equation. They have an interesting quaternionic analogue (cf. [A$_*$], [AV]). There are also the more general q-convex functions in complex analysis [HM], [S] and thier quaternionic analogues [HL$_{2,4,6}$].

There are many other general cases in which the outcome of Problem 1 is known and interesting. Some come from potential theory developed in calibrated and other geometries (cf. [HL$_{2,3}$]). Others come from universal subequations in riemannian geometry and on manifolds with topological G-structures (cf. [HL$_{6,7}$]). These will all be investigated here.

We begin the paper with definitions and a brief review of potential theory for fully nonlinear subequations. In order to introduce and motivate the restriction problem, we examine it for "geometrically determined subequations". These are subequations $F_{\mathbb{G}}$ determined by the condition $\mathrm{trace}\{\mathrm{Hess}\, u\big|_\xi\} \geq 0$ for all p-planes ξ in a given fixed subset $\mathbb{G} \subset G(p, \mathbf{R}^n)$ of the grassmannian of p-planes in $\mathbf{R}^n$. This, of course, includes the classical case of plurisubharmonic functions in complex analysis.

In Section 4 we prove the basic theorem of the paper. For a given subequation $F \subset J^2(Z)$ and submanifold $i : X \subset Z$, we formulate a *Restriction Hypothesis* and prove the following.

The Restriction Theorem 4.2. *Suppose $u \in \mathrm{USC}(Z)$. Assume that F satisfies the restriction hypothesis and suppose that i^*F_x is closed for each x. Then*

$$u \in F(Z) \qquad \Rightarrow \qquad u\big|_X \in (i^*F)(X).$$

This result is then applied throughout the rest of the paper.

In Section 5 we make some immediate but important applications. The first is the following. *Suppose $F \subset J^2(Z)$ is a translation-invariant, i.e., constant coefficient, subequation on an open set $Z \subset \mathbf{R}^n$. Then for $u \in \mathrm{USC}(Z)$,*

$$u \text{ is } F\text{-subharmonic on } Z \qquad \Rightarrow \qquad u\big|_X \text{ is } \overline{i^*F}\text{-subharmonic on } X.$$

Next we establish restriction for $\mathbb{G}$-plurisubharmonic functions, to affine $\mathbb{G}$-planes. Then we establish restriction for general linear subequations under a (necessary) linear restriction

hypothesis. This result becomes important in later applications. Finally, we examine restriction for first-order equations.

In Sections 6, 7 and 8 we establish quite general restriction theorems in the geometric setting. An example is the following. Let Z be a riemannian manifold of dimension n and $\mathbb{G} \subset G(p, TZ)$ a closed subset of the bundle of tangent p-planes on Z. Assume that $\mathbb{G} \subset G(p, TZ)$ admits a smooth neighborhood retraction which preserves the fibres of the projection $G(p, TZ) \to Z$. Then $\mathbb{G}$ determines a natural subequation F on Z defined by the condition that

$$\operatorname{trace}\left\{\operatorname{Hess} u\big|_{\xi}\right\} \ \geq\ 0 \quad \text{for all } \xi \in \mathbb{G}.$$

where $\operatorname{Hess} u$ denotes the riemannian hessian of u. (See (11.3) and [HL$_2$] for examples and details.) The corresponding F-subharmonic functions are again called **$\mathbb{G}$-plurisubhamronic functions**.

A **$\mathbb{G}$-submanifold** of Z is defined to be a p-dimensional submanifold $X \subset Z$ such that $T_xX \in \mathbb{G}$ for all $x \in X$.

THEOREM 7.1. *Let $X \subset Z$ be a $\mathbb{G}$-submanifold which is minimal (mean curvature zero). Then restriction to X holds for F. In other words, the restriction of any $\mathbb{G}$-plurisubharmonic function to X is subharmonic in the induced riemannian metric on X.*

In Section 8 this result is generalized to submanifolds of dimensions $> p$.

In Section 9 we formulate a quite different restriction result, based on the idea of jet equivalence. The notion of jet equivalence of subequations was introduced in [HL$_6$, §4] where it greatly extended the applicability of basic results. We recall that notion in this section and then refine it to the relative case. We then prove the following for an open subset $Z \subset \mathbf{R}^N$

THEOREM 10.1. *Let $i : X \hookrightarrow Z$ be an embedded submanifold and $F \subset J^2(Z)$ a subequation. Assume that F is locally jet equivalent modulo X to a constant coefficient subequation $\mathbf{F}$. If $\mathbf{H} \equiv i^*\mathbf{F}$ is a closed set, then $H \equiv i_X^* F$ is locally jet equivalent to the constant coefficient subequation $\mathbf{H}$, and restriction holds. That is,*

$$u \ \text{ is } F \text{ subharmonic on } \ Z \qquad \Rightarrow \qquad u\big|_X \ \text{ is } H \text{ subharmonic on } \ X$$

This theorem has a number of interesting applications. One is the following.

THEOREM 11.2. *Let Z be a riemannian manifold of dimension N and $F \subset J^2(Z)$ a subequation canonically determined by an O_N-invariant universal subequation $\mathbf{F} \subset \mathbf{J}^2_N$ (see §11). Then restriction holds for F on any totally geodesic submanifold $X \subset Z$.*

This result extends to subequations defined by G-invariant subsets of $\mathbf{R} \times \mathbf{R}^N \times \operatorname{Sym}^2(\mathbf{R}^N)$ on manifolds with topological G-structure.

Another application of Theorem 10.1 is to the study of potential theory on almost complex manifolds in the absence of any hermitian metric. It turns out that in this case there exists a well-defined subequation with the property that the corresponding subharmonic functions are exactly those whose restrictions to complex curves are subharmonic [HL$_8$]. The restriction theorem is central to this result.

In Appendix A we present some elementary examples where restriction fails.

In Appendix B certain important algebraic properties of restriction are studied.

Note. (Extension Theorems). Intimately related to restriction is the question of extension, namely, which functions on a submanifold can be extended to F-subharmonic functions in a neighborhood? In Appendix C we give conditions under which every C^2-function has this property.

Note. (The Addition Theorem). A second problem in the theory is that of addition. Suppose F and G are subequations on a domain $X \subset \mathbf{R}^n$. Then the closure of the fibre-wise sum $H = F + G$ is also a subequation, and obviously, if we have $u, v \in C^2(X)$ where u is F-subharmonic and v is G-subharmonic, then the sum $u + v$ is H-subharmonic. Extending this statement to functions $u, v \in \mathrm{USC}(X)$ is a deep and important question related to the comparison theorem.

The restriction results discussed here have a direct bearing on the addition problem. This can be summarized as follows. Suppose X and Y are euclidean domains with given subequations F and G respectively. Denote by $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ the projections onto the two factors, and consider the subequation on $X \times Y$ defined by taking the closure of

$$\mathbf{F} \ \equiv \ \pi_X^* F + \pi_Y^* G + \mathcal{P}_{X \times Y}.$$

Then the following is a reinterpretation of the Theorem on Sums (cf. [C, §11]).

Theorem on Sums. *If $u \in F(X)$ and $v \in G(Y)$, then the function $w(x, y) \equiv u(x) + v(y)$ satisfies $w \in \overline{\mathbf{F}}(X \times Y)$.*

We now set $X = Y$ and consider the diagonal embedding $\Delta : X \subset X \times X$. Note that the restricted subequation for $\mathbf{F}$ is $\Delta^* \mathbf{F} = H = F + G$. Applying the Restriction Theorem of Section 4 to this diagonal $\Delta(X) \subset X \times X$ leads to the desired addition theorem on X which is stated in Appendix C of [HL$_6$] (see [C, §10] where the technique is described without a general statement.)

2. Nonlinear Potential Theory

Suppose u is a real-valued function of class C^2 defined on an open subset $X \subset \mathbf{R}^n$. The **full second derivative** or **2-jet** of u at a point $x \in X$ will be denoted by

$$J_x u \ = \ \big(u(x), D_x u, D_x^2 u\big) \tag{2.1}$$

where $D_x u = (\frac{\partial u}{\partial x_1}(x), ..., \frac{\partial u}{\partial x_n}(x))$ and $D_x^2 u = ((\frac{\partial^2 u}{\partial x_i \partial x_j}(x)))$. Occasionally $D_x^2 u$ is denoted by $\mathrm{Hess}_x u$.

In this paper constraints on the full second derivative of a function $u \in C^2(X)$ will take the form

$$J_x u \ \in \ F_x \tag{2.2}$$

where $F \subset J^2(X)$ is a subset of the 2-jet space $J^2(X) = X \times \mathbf{R} \times \mathbf{R}^n \times \mathrm{Sym}^2(\mathbf{R}^n)$ and F_x denotes the fibre of F at $x \in X$. Such functions u will be called F**-subharmonic**.

Given an upper semi-continuous functions u on X with values in $[-\infty, \infty)$, a **test function for** u **at** x_0 is a C^2 function φ defined near x_0 which satisfies:

$$\left\{ \begin{array}{lll} u - \varphi & \le \ 0 & \text{near } x_0 \\ & = \ 0 & \text{at } x_0 \end{array} \right\}. \tag{2.3}$$

Definition 2.1. An upper semi-continuous function u on X is F**-subharmonic** if for all $x_0 \in X$

$$J_{x_0} \varphi \ \in \ F_{x_0} \quad \text{for all test functions } \varphi \text{ for } u \text{ at } x_0$$

Let $F(X)$ denote the space of all F-subharmonic functions on X.

Note that if $u(x_0) = -\infty$, then there are no test functions for u at x_0.

If φ is a test function for u at x_0, then so is $\psi \equiv \varphi + \frac{1}{2}\langle P(x - x_0), x - x_0\rangle$ for any matrix $P \ge 0$. Moreover, $J_{x_0}\psi = J_{x_0}\varphi + P$. Consequently, $F(X)$ is empty (except for $u \equiv -\infty$) unless F satisfies the following **positivity condition (P)**

$$F_x + \mathcal{P} \ \subset \ F_x \quad \text{for all } x \in X \tag{2.4}$$

where $\mathcal{P} \equiv \{0\} \times \{0\} \times \{P \in \mathrm{Sym}^2(\mathbf{R}^n) : P \ge 0\}$. We will abuse notation and also let $\mathcal{P}$ denote the subset of $\mathrm{Sym}^2(\mathbf{R}^n)$ of matrices $P \ge 0$.

Assuming this condition (P), it is easy to show that each C^2-function u satisfying (2.2) is F-subharmonic on X. (The converse is true without (P) since $\varphi = u$ is a test function.)

Definition 2.1 can be recast in a more useful form. (See [HL_6, Prop. A.1 (IV)].)

Lemma 2.2. *Suppose $F \subset J^2(X)$ is a closed subset, and let u be an upper semi-continuous function on X. Then $u \notin F(X)$ if and only if $\exists\, x_0 \in X$, $\alpha > 0$ and $(r, p, A) \notin F_{x_0}$ with*

$$\begin{array}{llll} u(x) - \big[r + \langle p, x - x_0\rangle + \frac{1}{2}\langle A(x - x_0), x - x_0\rangle\big] & \le \ -\alpha|x - x_0|^2 & \text{near } x_0 & \textit{and} \\ & = \ 0 & \textit{at } \ x_0 & \end{array}$$

Using this Lemma, basic potential theory for F-subharmonic functions is elementary to establish. See Appendices A and B in [HL$_6$].

THEOREM 2.3. *Let F be an arbitrary closed subset of $J^2(X)$.*

(A) (Local Property) u is locally F-subharmonic if and only if u is globally F-subharmonic.

(B) (Maximum Property) If $u, v \in F(X)$, then $w = \max\{u, v\} \in F(X)$.

(C) (Coherence Property) If $u \in F(X)$ is twice differentiable at $x \in X$, then $j^2_x u \in F_x$.

(D) (Decreasing Sequence Property) If $\{u_j\}$ is a decreasing ($u_j \geq u_{j+1}$) sequence of functions with all $u_j \in F(X)$, then the limit $u = \lim_{j\to\infty} u_j \in F(X)$.

(E) (Uniform Limit Property) Suppose $\{u_j\} \subset F(X)$ is a sequence which converges to u uniformly on compact subsets to X, then $u \in F(X)$.

(F) (Families Locally Bounded Above) Suppose $\mathcal{F} \subset F(X)$ is a family of functions which are locally uniformly bounded above. Then the upper semicontinuous regularization $u = v^*$ of the upper envelope

$$v(x) \;=\; \sup_{f\in\mathcal{F}} f(x)$$

belongs to $F(X)$.

There are certain obvious additional properties (e.g. *If $F_1 \subset F_2$, then $u \in F_1(X) \;\Rightarrow\; u \in F_2(X)$*), which will be used without reference.

Although the positivity condition (P) is not needed in the proofs of either Lemma 2.2 or Theorem 2.3, the fact that without (P) there are no F-subharmonic functions, other than $u \equiv -\infty$, explains its usefulness.

Definition 2.4. A closed subset $F \subset J^2(X)$ which satisfies the positivity condition (P) will be called a **subequation**.

Note. This does not agree with the terminology of [HL$_6$] where subequations were assumed to have two additional properties: a stronger topological condition (T) and, in order to have a chance of proving uniqueness in the Dirichlet problem, standard negativity condition (N) on the values of the dependent variable (cf. [HL$_6$]). These conditions are unnecessary for the discussion here.

3. An Introduction to Restriction – The Geometric Case.

In this section we describe a restriction theorem in a simple but important case. A subequation F is said to be **geometrically determined** by a closed subset $\mathbb{G}$ of the Grassmannian $G(p, \mathbf{R}^n)$ of (unoriented) p-planes through the origin in $\mathbf{R}^n$ if $F \equiv F_{\mathbb{G}}$ is defined by

$$\text{trace}\left\{D_x^2 u\big|_W\right\} \ \geq \ 0 \ \text{ for all } W \in \mathbb{G} \tag{3.1}$$

and for all $x \in X$. The upper semi-continuous functions in $F_{\mathbb{G}}(X)$ will be referred to as $\mathbb{G}$-plurisubharmonic on X.

Example 3.1. (Classical Subharmonicity). If $p = n$ and $\mathbb{G} = G(n, \mathbf{R}^n) = \{\mathbf{R}^n\}$, then u is $\mathbb{G}$-plurisubharmonic on the open set $X \subset \mathbf{R}^n$ if and only if u is subharmonic (trace$(D^2u) = \Delta u \geq 0$ in the C^2-case) using any of the equivalent classical definitions ($u \equiv -\infty$ on components of X is allowed). In the case $n = 1$, subharmonicity is the same as classical convexity in one variable, expanded to allow $u \equiv -\infty$ as a matter of convenience.

An **affine $\mathbb{G}$-plane** is an affine plane in $\mathbf{R}^n$ whose translate through the origin belongs to $\mathbb{G}$.

Restriction Theorem 3.2. *A function u is $\mathbb{G}$-plurisubharmonic on $U \subset \mathbf{R}^n$ if and only if*

$$u\big|_{U\cap W} \ \text{ is subharmonic for each affine } \mathbb{G}\,\text{plane } W. \tag{3.2}$$

Proof. Half of the proof is trivial. If φ is a test function for u at $x_0 \in X$ with $J_{x_0}\varphi \notin F_{x_0}$, then by definition of $F \equiv F_{\mathbb{G}}$ there exists a $W \in \mathbb{G}$ with $\text{tr}_W D^2_{x_0}\varphi < 0$. Therefore (cf. Ex. 3.1) $u\big|_{X\cap(W+x_0)}$ is not subharmonic at x_0. The other half, namely the assertion that restrictions of $\mathbb{G}$-psh functions to affine $\mathbb{G}$-planes are subharmonic is proved in the Section 5. It is a special case of Theorem 5.3. ∎

Example 3.3. (Classical Convexity). If $\mathbb{G} = G(1, \mathbf{R}^n)$, then this restriction theorem is precisely the theorem required to establish that the condition $D^2u \geq 0$ in the viscosity sense is equivalent to u being convex (or possibly $\equiv -\infty$). Somewhat surprisingly we were unable to find an elementary viscosity proof of this fact in the literature. Such a proof is essentially given in [HL$_4$, Prop. 2.6], and this is the prototype of our proof of the general restriction theorem.

Example 3.4. (Plurisubhamonicity in Complex Analysis). A function $u \in \text{USC}(X)$ with X an open subset of $\mathbf{C}^n$ is said to be plurisubharmonic if the restriction of u to each affine complex line is classically subhharmonic. Our Restriction Theorem 3.2 states that this classical notion is equivalent to being $\mathbb{G}$-plurisubharmonic where $\mathbb{G} = G_{\mathbf{C}}(1, \mathbf{C}^n) \subset G_{\mathbf{R}}(2, \mathbf{C}^n)$ is the Grassmannian of complex lines in $\mathbf{C}^n$.

Further examples abound. A wide class (including Examples 3.3 and 3.4) is given by choosing a calibration $\phi \in \Lambda^p\mathbf{R}^n$ and then setting

$$\mathbb{G}(\phi) \ \equiv \ \left\{W \in G(p, \mathbf{R}^n) : \phi\big|_W \text{ is the standard volume form on } W\right\} \tag{3.3}$$

for one of the choices of orientation on W.

4. The Restriction Theorem.

Suppose Z is an open subset of $\mathbf{R}^N = \mathbf{R}^n \times \mathbf{R}^m$ with coordinates $z = (x, y)$. Set $X = \{x \in \mathbf{R}^n : (x, y_0) \in Z\}$ for a fixed y_0, and let $i : X \hookrightarrow Z$ denote the inclusion map $i(x) = (x, y_0)$. Adopt the notation

$$r = \varphi(x, y_0),\ p = \frac{\partial\varphi}{\partial x}(x, y_0),\ q = \frac{\partial\varphi}{\partial y}(x, y_0),\ A = \frac{\partial^2\varphi}{\partial x^2}(x, y_0),\ B = \frac{\partial^2\varphi}{\partial y^2}(x, y_0),\ C = \frac{\partial^2\varphi}{\partial x\partial y}(x, y_0)$$

for the 2-jet $J_z\varphi$ of a function φ at $z = (x, y_0)$. Then the 2-jet of th restricted function $\psi(x) = \varphi(x, y_0)$ is given by $J_x\psi = (r, p, A)$. Thus, restriction $i^* : J^2(Z) \longrightarrow J^2(X)$ on 2-jets is given by

$$i^*\left(r,\ (p, q),\ \begin{pmatrix} A & C \\ C^t & B \end{pmatrix}\right) \quad = \quad (r, p, A) \qquad \text{at}\ \ i(x) = z. \tag{4.1}$$

If F is a subset of $J^2(Z)$, then the **restriction of F to X** is the subset $H \subset J^2(X)$ defined by

$$H_x \ = \ i^* F_{i(x)} \ \ \text{for all}\ x \in X. \tag{4.2}$$

Each quadratic form $P \geq 0$ on $\mathbf{R}^n$ is the restriction of a quadratic form $\widetilde{P} \geq 0$ on $\mathbf{R}^N$. This proves that:

$$\text{If } F \text{ satisfies the positivity condition (P), then } H = i^*F \ \text{ also satisfies (P).} \tag{4.3}$$

We shall also consider the fibre-wise closure $\overline{H}$ of H, that is $\overline{H}_x = \overline{i^* F_{i(x)}}$. It is obvious that

$$F \ \text{ satisfies } \ (\mathrm{P}) \quad \Rightarrow \quad \overline{H} \ \text{ satisfies } \ (\mathrm{P}) \tag{4.3$'$}$$

Definition 4.1. We say that **restriction to X holds for F** if

$$u \text{ is } F\text{−subharmonic on } Z \quad \Rightarrow \quad u\big|_X \text{ is } H\text{−subharmonic on } X \tag{4.4}$$

This is not always the case. Some elementary examples are presented in Appendix A. Of course, if $u \in C^2(Z)$ is F-subharmonic, then $u\big|_X$ is H-subharmonic on X since $i^*Ju = Ji^*u$. The only issue is with $u \in \mathrm{USC}(Z)$ that are not C^2.

The Restriction Hypothesis: Given $x_0 \in X$ and $(r_0, p_0, A_0) \in \mathbf{J}^2_n$ and given $z_\epsilon = (x_\epsilon,\, y_\epsilon)$ and r_ϵ for a sequence of real numbers ϵ converging to 0.

$$\text{If} \quad \left(r_\epsilon,\ \left(p_0 + A_0(x_\epsilon - x_0),\ \frac{y_\epsilon - y_0}{\epsilon}\right),\ \begin{pmatrix} A_0 & 0 \\ 0 & \frac{1}{\epsilon} I \end{pmatrix}\right) \ \in \ F_{z_\epsilon} \tag{4.5}$$

$$\text{and} \ \ x_\epsilon \ \to \ x_0, \quad \frac{|y_\epsilon - y_0|^2}{\epsilon} \ \to \ 0, \quad r_\epsilon \ \to \ r_0, \tag{4.6}$$

then

$$(r_0, p_0, A_0) \ \in \ H_{x_0}.$$

Remark. If the subequation F is independent of the r-variable, that is, if F_x can be considered as a subset of the reduced 2-jet space $\overline{\mathbf{J}}_x^2 = \mathbf{R}^n \times \mathrm{Sym}^2(\mathbf{R}^n)$, then the restriction hypothesis can be restated as follows.

Restriction Hypothesis (Second Version – for r-Independent Subequations): Given $x_0 \in X$ and $z_\epsilon = (x_\epsilon, y_\epsilon)$ converging to $z_0 = (x_0, y_0)$ with $\frac{1}{\epsilon}|y_\epsilon - y_0|^2 \to 0$, for a sequence of real numbers ϵ converging to 0, consider the polynomials

$$\psi_\epsilon(x,y) \;\equiv\; r_0 + \langle p_0, x - x_0\rangle + \tfrac{1}{2}\langle A_0(x-x_0), x - x_0\rangle + \frac{1}{2\epsilon}|y - y_0|^2. \tag{4.7}$$

$$\text{If} \quad \overline{J}_{z_\epsilon}\psi_\epsilon \in F_{z_\epsilon} \quad \text{for all } \epsilon, \quad \text{then} \quad (p_0, A_0) \in H_{z_0}$$

This follows since the reduced jet $\overline{J}_{z_\epsilon}\psi_\epsilon$ equals the jet in (4.5) modulo $r_\epsilon - r_0$.

The Restriction Theorem 4.2. *Suppose $u \in \mathrm{USC}(Z)$. Assume the restriction hypothesis and suppose that H_x is closed for each x. Then*

$$u \in F(Z) \qquad \Rightarrow \qquad u\big|_X \in H(X).$$

Remark 4.3. The assumption that H_x is closed can be dropped if the conclusion is weakened to

$$u \in F(Z) \qquad \Rightarrow \qquad u\big|_X \in \overline{H}(X).$$

Proof. If $u\big|_X \notin H(X)$, then by Lemma 2.2 (since H_x is closed) there exists $x_0 \in X$, $\alpha > 0$, and $(r_0, p_0, A_0) \notin H_{x_0}$ such that

$$\begin{aligned} u(x, y_0) - Q(x) \;&\le\; -\alpha|x - x_0|^2 \qquad \text{near } x_0 \qquad \text{and} \\ &=\; 0 \qquad\qquad\qquad\quad \text{at } \; x_0 \end{aligned} \tag{4.8}$$

where

$$Q(x) \;\equiv\; r_0 + \langle p_0, x - x_0\rangle + \tfrac{1}{2}\langle A_0(x - x_0), x - x_0\rangle. \tag{4.9}$$

In the next step we construct $z_\epsilon = (x_\epsilon, y_\epsilon)$ satisfying (4.6) with $r_\epsilon \equiv u(z_\epsilon)$. Set

$$w(x,y) \;\equiv\; u(x,y) - Q(x).$$

Let $B(z_0)$ denote a small closed ball about z_0 in $\mathbf{R}^N$, so that (4.8) holds on the y_0-slice. For each $\epsilon > 0$ small, let

$$M_\epsilon \;\equiv\; \sup_{B(z_0)} \left(w - \tfrac{1}{2\epsilon}|y - y_0|^2\right), \tag{4.10}$$

and choose z_ϵ to be a maximum point. Since the value of this function at z_0 is zero, the maximum value $M_\epsilon \ge 0$. Furthermore, the M_ϵ decrease to a limit, say M_0. Now

$$\begin{aligned} M_\epsilon \;&=\; w(z_\epsilon) - \tfrac{1}{2\epsilon}|y_\epsilon - y_0|^2 \;=\; w(z_\epsilon) - \tfrac{1}{4\epsilon}|y_\epsilon - y_0|^2 - \tfrac{1}{4\epsilon}|y_\epsilon - y_0|^2 \\ &\le M_{2\epsilon} - \tfrac{1}{4\epsilon}|y_\epsilon - y_0|^2, \qquad \text{that is} \end{aligned}$$

$$\tfrac{1}{4\epsilon}|y_\epsilon - y_0|^2 \ \le\ M_{2\epsilon} - M_\epsilon.$$

Thus

$$\tfrac{1}{\epsilon}|y_\epsilon - y_0|^2 \ \longrightarrow\ 0 \tag{4.11}$$

and in particular $y_\epsilon \to y_0$.

Suppose now that $\bar z = (\bar x, y_0)$ is a cluster point of $\{z_\epsilon\}$. Then taking a sequence $z_\epsilon \to \bar z$

$$M_0 \ =\ \lim_{\epsilon\to 0} M_\epsilon \ =\ \lim_{\epsilon\to 0}(w(z_\epsilon) - \tfrac{1}{2\epsilon}|y_\epsilon - y_0|^2) \ =\ \lim_{\epsilon\to 0} w(z_\epsilon) \ \le\ w(\bar z) \tag{4.12}$$

by (4.10), (4.11) and the fact that w is upper semi-continuous. By (4.8) and the fact that $\bar y = y_0$, we have $w(\bar z) \le 0$. Hence, $M_0 = w(\bar z) = 0$. Since $w(x, y_0)$ has a strict maximum of 0 at $z_0 = (x_0, y_0)$, and this maximum value is attained at $\bar z = (\bar x, y_0)$, we must have $\bar x = x_0$. Thus

$$x_\epsilon \ \to\ x_0. \tag{4.13}$$

Now by (4.12), we have $0 = \lim_{\epsilon\to 0} w(z_\epsilon) = \lim_{\epsilon\to 0}\big(u(z_\epsilon) - Q(z_\epsilon)\big) = \lim_{\epsilon\to 0} r_\epsilon - r_0$, which completes the proof that (4.6) is satisfied.

It remains to verify (4.5). The notation has been arranged so that

$$u - \psi_\epsilon \ =\ w - \frac{1}{2\epsilon}|y - y_0|^2 \tag{4.14}$$

where ψ_ϵ is defined by (4.7). Consequently, (4.10) can be restated as

$$\begin{aligned} u - \psi_\epsilon \ &\le\ M_\epsilon \qquad \text{near } z_\epsilon \qquad \text{and} \\ &=\ M_\epsilon \qquad \text{at } \ z_\epsilon, \end{aligned} \tag{4.10$'$}$$

that is, $\varphi_\epsilon \equiv \psi_\epsilon + M_\epsilon$ is a test function for u at z_ϵ. This implies that $j^2_{z_\epsilon}\varphi_\epsilon \in F_{z_\epsilon}$. Computing this 2-jet verifies (4.5). The Restriction Hypothesis now implies that $(r_0, p_0, A_0) \in H_{x_0}$, which is a contradiction. ∎

5. First Applications.

We now examine some applications of the Restriction Theorem 4.2

Restriction in the Constant Coefficient Case

Suppose $F = Z \times \mathbf{F}$ for $\mathbf{F} \subset \mathbf{J}^2_N$. Then F is said to have **constant coefficients** on Z. Now consider $X = Z \cap \{y = y_0\}$ as above. If F has constant coefficients on Z, then the restriction of 2-jets gives a set $H = i^* F = X \times \mathbf{H}$ with constant coefficients on X. Note that $\overline{H} = X \times \overline{\mathbf{H}}$ also has constant coefficients.

THEOREM 5.1. *Suppose $F \subset J^2(Z)$ is closed, has constant coefficients and satisfies (P). Then*

$$u \text{ is } F\text{-subharmonic on } Z \qquad \Rightarrow \qquad u\big|_X \text{ is } \overline{H}\text{-subharmonic on } X.$$

Proof. In this case the restriction hypothesis is easy to verify. Since

$$\left(r_\epsilon, (p_\epsilon, q_\epsilon), \begin{pmatrix} A_0 & 0 \\ 0 & \frac{1}{\epsilon} I \end{pmatrix}\right) \in F_{z_e} = \mathbf{F},$$

we have that the 2-jet $(r_\epsilon, p_\epsilon, A_0) \in H$ even though $z_\epsilon \notin X$. Now the fact that $r_\epsilon \to r_0$ and $p_\epsilon = p_0 + A_0(x_\epsilon - x_0) \to p_0$ is enough to conclude that $(r_0, p_0, A_0) \in \overline{\mathbf{H}} = \overline{H}_{z_0}$. ∎

There are many subequations for which Theorem 5.1 is interesting. One class of examples comes from considering the ordered eigenvalues $\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$ of a symmetric matrix $A \in \mathrm{Sym}^2(\mathbf{R}^N)$. With standard coordinates (r, p, A) for $\mathbf{J}^2_N$ we define

$$\mathbf{P}_k \equiv \{\lambda_k(A) \geq 0\} \subset \mathbf{J}^2_N$$

and consider the constant coefficient subequation $\mathcal{P}_k(Z) = Z \times \mathbf{P}_k$. Note that $\lambda_k(A) = \inf_W \{\sup_w \langle Aw, w \rangle\}$ where the inf is taken over linear subspaces $W \subset \mathbf{R}^N$ of dimension k and the sup is over unit vectors $w \in W$. From this one see that

$$i^* \mathcal{P}_k(Z) = \mathcal{P}_k(X).$$

The analogous observation holds for subquations given by $\{\lambda_k(A) \geq f(r, |p|)\}$ for a general function f.

Example 5.2. (Restriction Fails). Define F on $\mathbf{R}^2$ by $|p||q| \geq 1$. Then the restricted subequation H on $\{y = 0\}$ is defined by $p \neq 0$. In particular, H is not closed. Taking $p_0 = 0$, $A_0 = I$, and $x_\epsilon - x_0 \equiv p_\epsilon \to 0$, we see that the restriction hypothesis is false. To see that restriction fails consider $u(x, y) = |y|^\alpha e^x + \frac{1}{3}x^3$. It is easy to verify that u is F-subharmonic near the origin in $\mathbf{R}^2$ if $0 < 2\alpha < 1$. However, the restricted function $u(x, 0) = \frac{1}{3}x^3$ is not H-subharmonic near $x = 0$. Note that $\overline{H}$ is all of J^2 so that the conclusion of Theorem 5.1 is trivial.

The Geometric Case.

As in Section 3 suppose that $F_{\mathbb{G}}$ is geometrically defined by closed subset $\mathbb{G}$ of the grassmannian $G(p, \mathbf{R}^N)$. We note that Theorem 3.2 is a special case of Theorem 5.1. To see this suppose $\mathbf{W}$ is an affine $\mathbb{G}$-plane with (constant) tangent plane $W \in \mathbb{G}$. Then for any quadratic form Q at any point of $\mathbf{W}$ we have $\mathrm{tr}_W i_{\mathbf{W}}^* Q = \mathrm{tr}_W Q$ which proves that $i_{\mathbf{W}}^* \mathbf{F}_{\mathbb{G}} \subset \mathbf{F}_{\{W\}}$, the classical (subharmonic) subequation on $\mathbf{W}$ (cf. Example 3.1).

This Restriction Theorem 3.2 can be generalized by considering a subspace $V \subset \mathbf{R}^N$ of dimension $n \geq p$ and defining

$$\mathbb{G}(V) \ \equiv \ \{W \in \mathbb{G} : W \subset V\} \tag{5.1}$$

to be the space of $\mathbb{G}$-planes which are **tangential to** V. Since $\mathbb{G}(V)$ is a closed subset of the grassmannian $G(p, \mathbf{R}^N)$, it geometrically determines a subequation $F_{\mathbb{G}(V)}$ on V by

$$F_{\mathbb{G}(V)} \ \equiv \ \{a \in \mathrm{Sym}^2(V^*) : \mathrm{tr}_W a \geq 0 \ \ \forall\, W \in \mathbb{G}(V)\} \tag{5.2}$$

THEOREM 5.3. *If u is $\mathbb{G}$-plurisubharmonic on an open subset $U \subset \mathbf{R}^N$, then for each affine subspace V of $\mathbf{R}^N$,*

$$u\big|_{U \cap V} \ \text{ is } \ \mathbb{G}(V) \text{ plurisubharmonic.}$$

Theorem 3.2 is the special case where $V = W$ and so $\mathbb{G}(V) = \{W\}$.

Remark 5.4. As in Theorem 3.2 the converse (where one considers all affine subspaces V of dimension n with $n \geq p$) is trivial.

Proof. Let i_V^* denote the restriction of 2-jets from $\mathbf{R}^N$ to $V = \mathbf{R}^n$. For $W \subset V$ one has $\mathrm{tr}_W i_V^* Q = \mathrm{tr}_W Q$ for all quadratic forms Q, which proves that

$$i_V^* F_{\mathbb{G}} \ \subset \ F_{\mathbb{G}(V)}. \tag{5.3}$$

Therefore $\overline{H} = \overline{i_V^* F_{\mathbb{G}}} \subset F_{\mathbb{G}(V)}$, and so Theorem 5.3 is a special case of Theorem 5.1. ∎

In Appendix B we prove that in fact

$$i_V^* F_{\mathbb{G}} = F_{\mathbb{G}(V)}.$$

Subequations which can be Defined Using Fewer of the Variables in $\mathbf{R}^N$.

Suppose that F can be defined using fewer of the variables in $\mathbf{R}^N$, say using only the variables in $\mathbf{R}^n \subset \mathbf{R}^N$. This means by definition that there exists $\mathbf{H} \subset \mathbf{J}_n^2$ with $\mathbf{F} = i^* \mathbf{H}$ where $i^* : \mathbf{J}_N^2 \to \mathbf{J}_n^2$ is the restriction map.

Definition 5.5. A function $u \in \mathrm{USC}(Z)$ is *horizontally H-subharmonic on an open set* $Z \subset \mathbf{R}^N$ if for each $y_0 \in \mathbf{R}^m$ the function $u(x, y_0)$ is of type H on $Z \cap \{y = y_0\}$.

As a special case of Theorem 5.1 we have

THEOREM 5.6. *Suppose the constant coefficient subequation $\mathbf{F} = (i^*)^{-1}(\mathbf{H})$ can be defined using the variables $\mathbf{R}^n \subset \mathbf{R}^N$. Then u is F-subharmonic on Z if and only if u is horizontally H-subharmonic on Z.*

Restriction in the Linear Case

Consider the second-order linear operator with smooth coefficients:

$$\mathbb{L}\left(z, r, (p,q), \begin{pmatrix} A & C \\ C & B \end{pmatrix}\right) \;\equiv\; \langle a(z), A\rangle + \langle \alpha(z), p\rangle + \gamma(z)r + \langle b(z), B\rangle + \langle \beta(z), q\rangle + \langle c(z), C\rangle$$

Let $L \subset Z \times \mathbf{J}^2_N$ be the subset defined by $\mathbb{L} \geq 0$, and consider $H_x \equiv i^* L_z$ with $z = (x, y_0) \in X$.

We will prove that restriction holds in two cases, which taken together "essentially" exhaust linear operators. In the first case we assume that at least on of the coefficients $\beta(x_0, y_0), b(x_0, y_0)$ or $c(x_0, y_0)$ in non-zero. Restriction locally holds but is completely trivial since $H_x = \mathbf{J}^2_n$ is everything for x near x_0. If, for example, $\beta(x_0, y_0) \neq 0$, then by choosing q to be a sufficiently large multiple of $\beta(x_0, y_0)$, any jet (r, p, A) can be shown to lie in H_x. The second case is much more interesting. We assume the following **linear restriction hypothesis**:

$$\beta(x, y_0), b(x, y_0), \text{ and } c(x, y_0) \text{ vanish identically on } X \tag{5.4}$$

Define the linear operator

$$\mathbb{L}_X(x, r, pA) \;\equiv \langle a(x, y_0), A\rangle + \langle a(x, y_0), p\rangle + \gamma(x, y_0)r \tag{5.5}$$

on X. Under this hypothesis H is the subset of $X \times \mathbf{J}^2_n$ defined by $\mathbb{L}_X \geq 0$.

THEOREM 5.7. *Assume that L satisfies Condition (P) and the linear restriction hypothesis. Then*

$$u \text{ is } \mathbb{L}\text{-subharmonic on } Z \qquad \Rightarrow \qquad u\big|_X \text{ is } \mathbb{L}_X\text{-subharmonic on } X.$$

Proof. Since β vanishes on X, we have $|\beta(x,y)| \leq C|y - y_0|$. Moreover, since b vanishes on X and since (P) implies $b(z) \geq 0$, b must vanish to second order, i.e., $|b(x,y)| \leq C|y - y_0|^2$. These two facts are enough to verify the restriction hypothesis in Lemma 4.1. Assume that

$$\begin{aligned} 0 \;\leq\; & \mathbb{L}\left(z_\epsilon, r_\epsilon, (p_0 + A_0(x_\epsilon - x_0), \tfrac{y_\epsilon - y_0}{\epsilon}), \begin{pmatrix} A_0 & 0 \\ 0 & \frac{1}{\epsilon}I \end{pmatrix}\right) \\ =\; & \langle a(z_\epsilon), A_0\rangle + \langle \alpha(z_\epsilon), p_0 + A_0(x_\epsilon - x_0)\rangle + \gamma(z_\epsilon)r_\epsilon + \langle b(z_\epsilon), \tfrac{1}{\epsilon}I\rangle + \langle \beta(z_\epsilon), \tfrac{y_\epsilon - y_0}{\epsilon}\rangle \end{aligned}$$

and that

$$x_\epsilon \;\to\; x_0, \quad \frac{|y_\epsilon - y_0|^2}{\epsilon} \;\to\; 0, \quad \text{and} \quad r_\epsilon \;\to\; r_0.$$

Now

$$\left|\left\langle \beta(z_\epsilon), \frac{y_\epsilon - y_0}{\epsilon}\right\rangle\right| \ \leq\ C\,\frac{|y_\epsilon - y_0|^2}{\epsilon} \ \to\ 0 \quad \text{and}$$

$$\left|\left\langle b(z_\epsilon), \frac{1}{\epsilon} I\right\rangle\right| \ \leq\ C\,\frac{|y_\epsilon - y_0|^2}{\epsilon} \ \to\ 0.$$

Hence the RHS converges to

$$\langle a(z_0), A_0\rangle + \langle \alpha(z_0), p_0\rangle + \gamma(z_0) r_0 \ =\ \mathbb{L}_X(z_0, r_0, p_0 A_0)$$

which proves that $(z_0, r_0, p_0, A_0) \in H_{x_0}$. ∎

Remark 5.8 (Versions of the Linear Restriction Hypothesis). The following conditions are equivalent. The first is (5.4) above.

(1) $b(x, y_0)$, $\beta(x, y_0)$ and $c(x, y_0)$ vanish on X.

(2) H is the subset $\{\mathbb{L}_X \geq 0\}$ of $X \times \mathbf{J}^2_n$

(3) $(\mathbb{L} f)(x, y_0) = \mathbb{L}_X(f(x, y_0))$ for all smooth functions f on Z.

(3)$'$ There exists an intrinsic operator $\mathbb{L}'_X$ on X such that $(\mathbb{L} f)(x, y_0) = \mathbb{L}'_X(f(x, y_0))$ for all smooth functions f on Z.

(4) $L_{i(x)} \ =\ (i^*)^{-1}(H_x) \quad \forall\, x \in X$.

First Order Restriction

Suppose F is *first order*, that is, F is a subset of $Z \times \mathbf{J}^1_N$. By convention the *F-subharmonic functions on Z* are the same thing as the subharmonic functions for the set $F \times \mathrm{Sym}^2(\mathbf{R}^n) \subset \mathbf{J}^2_N$. If for all compact $K \subset Z$ and $R > 0$,

$$\{(x, r, p) \in F : x \in K, |r| \leq R\} \ \text{ is compact,} \tag{5.6}$$

then F is said to be *coercive*.

If $i : X \hookrightarrow Z$ is defined by $i(x) = (x, y_0)$, and $H_x \equiv i^* F$ where i^* is restriction of 1-jets, then

$$H_x \ =\ \{(r, p) : \exists\, q \text{ with } (r, (p, q)) \in F_{i(x)}\} \tag{5.7}$$

If F is coercive, then H is coercive.

THEOREM 5.9. *If $F \subset J^1(Z)$ is coercive and $i : X \hookrightarrow Z$ is defined by $i(x) = (x, y_0)$, then*

$$u \in F(Z) \qquad \Rightarrow \qquad u\big|_X \in H(X).$$

Proof. The Restriction Hypothesis is easy to verify in this case. Given $z_0 \in X$ and (r_0, p_0, A_0), if

$$z_\epsilon \to z_0, \quad r_\epsilon \to r_0, \quad \text{and} \quad (r_\epsilon, (p_0 + A_0(x_\epsilon - x_0), \tfrac{1}{\epsilon}(y_\epsilon - y_0)) \ \in\ F_{z_\epsilon},$$

then by the coerciveness of F we can extract a subsequence $(z_\epsilon, r_\epsilon, (p_\epsilon, q_\epsilon))$ which converges to $(z', r', (p', q')) \in F_{z_0}$. (Here $p_\epsilon \equiv p_0 + A_0(x_\epsilon - x_0)$ and $q_\epsilon \equiv \frac{1}{\epsilon}(y_\epsilon - y_0)$.) But $z' = z_0$, $r' = r_0$, and $p' = p_0$. Hence $(r_0, p_0) \in H_{x_0}$, which proves the Restriction Hypothesis. ∎

6. The Geometric Case – Restriction to Minimal $\mathbb{G}$-Submanifolds.

Recall the "geometric case" where the subequation F is determined by a closed subset $\mathbb{G}$ of the Grassmannian $G(p, \mathbf{R}^n)$ from section 3. In this section the Restriction Theorem 3.2 will be generalized in two ways.

First , "the coefficients of the subequation are allowed to vary". That is, a closed subset $\mathbb{G} \subset X \times G(p, \mathbf{R}^n)$ is given with fibres $\mathbb{G}_x \subset G(p, \mathbf{R}^n)$, and the subequation F with fibres F_x is defined by the condition

$$\operatorname{trace}\left(A\big|_W\right) \ \geq\ 0 \ \text{ for all } W \in \mathbb{G}_x. \tag{3.1$'$}$$

As before, we say that F is **geometrically determined** by $\mathbb{G} \subset X \times G(p, \mathbf{R}^n)$.

Second, the affine $\mathbb{G}$-planes in the RestrictionTheorem 3.2 are replaced by $\mathbb{G}$-submanifolds with mean curvature zero.

Definition 6.1. A p-dimensional submanifold M of $X \subset \mathbf{R}^n$ is a **$\mathbb{G}$-submanifold** if $T_xM \in \mathbb{G}_x$ for each $x \in M$.

THEOREM 6.2. *Suppose u is a $\mathbb{G}$-plurisubharmonic function on X and M is a $\mathbb{G}$-submanifold of X which is minimal. Further assume that $\mathbb{G} \subset X \times G(p, \mathbf{R}^n)$ has a smooth neighborhood retract which preserves the fibres $\{x\} \times G(p, \mathbf{R}^n)$. If u is $\mathbb{G}$-plurisubharmonic on X, then $u\big|_M$ is Δ_M-subharmonic, where Δ_M is the Laplace-Beltrami operator for the induced metric on M.*

Proof. The conclusion is local. Choose a local orthonormal frame field $e_1, ..., e_p$ on M and extend it to an orthonormal frame field $e_1, ..., e_p$ in a neighborhood U in $\mathbf{R}^n$. Define

$$W(x) \ = \ \rho(\operatorname{span}\{e_1(x), ..., e_p(x)\})$$

where ρ is the neighborhood retract onto $\mathbb{G}$. Then $W(x)$ defines a linear operator

$$(\mathbb{L}f)(x) \ \equiv\ \langle P_{W(x)}, \operatorname{Hess}_x f\rangle, \quad \text{for} \ \ f \in C^\infty(U) \tag{6.1}$$

(where P_W denotes orthogonal projection onto W). Since each $W(x) \in \mathbb{G}$, we see that if f is $\mathbb{G}$-plurisubharmonic, then f is $\mathbb{L}$-subharmonic. Since $W(x) = T_xM$ for all $x \in M$ we have

$$(\mathbb{L}f)(x) \ = \ \langle T_xM, \operatorname{Hess}_x f\rangle \ = \ (\Delta_M f)(x) + (H_M f)(x) \qquad \forall\, x \in M$$

where H_M is the mean curvature vector field of M (see [HL$_2$] for example). Since M is a minimal submanifold, this proves that

$$(\mathbb{L}f)(x) \ = \ (\Delta_M f)(x) \qquad \forall\, x \in M \text{ and } f \in C^\infty(U). \tag{6.2}$$

Now make a coordinate change so that M becomes $X = \mathbf{R}^p \times \{0\} \subset \mathbf{R}^p \times \mathbf{R}^{n-p}$. By (3)$'$ in Remark 5.8 the linear restriction hypothesis is satisfied. Therefore Theorem 5.7 implies that if an u.s.c. function u is $\mathbb{G}$-plurisubharmonic, then $u\big|_M$ is Δ_M-subharmonic. ∎

Remark 6.3. Here we used the obvious fact that $F_1 \subset F_2 \quad \Rightarrow \quad F_1(X) \subset F_2(X)$ to conclude that if u is $\mathbb{G}$-plurisubharmonic, then u is $\mathbb{L}$-subharmonic.

7. Geometric Examples on Riemannian Manifolds.

The result of the last section can be carried over to a much more general context. Let Z be a riemannian manifold of dimension n and $\mathbb{G} \subset G(p, TZ)$ a closed subset of the bundle of tangent p-planes on Z. We again assume that $\mathbb{G} \subset G(p, TZ)$ admits a smooth neighborhood retraction which preserves the fibres of the projection $G(p, TZ) \to Z$. As before $\mathbb{G}$ determines a natural subequation F on Z defined by the condition that

$$\operatorname{trace}\left\{\operatorname{Hess} u\big|_{\xi}\right\} \;\geq\; 0 \quad \text{for all } \xi \in \mathbb{G}.$$

where $\operatorname{Hess} u$ denotes the riemannian hessian of u. (See [$\mathrm{HL}_{2,6}$] for examples and details.) The corresponding F-subharmonic functions are again called **$\mathbb{G}$-plurisubhamronic functions**.

By a **$\mathbb{G}$-submanifold** of Z we mean a p-dimensional submanifold $X \subset Z$ such that $T_xX \in \mathbb{G}$ for all $x \in X$. The following result generalizes a basic theorem in [HL_5]* for C^2-functions to general upper semi-continuous $\mathbb{G}$-plurisubharmonic functions.

THEOREM 7.1. *Let $X \subset Z$ be a $\mathbb{G}$-submanifold which is minimal (mean curvature zero). Then restriction to X holds for F. In other words, the restriction of any $\mathbb{G}$-plurisubharmonic function to X is subharmonic in the induced riemannian metric on X.*

Proof. Choose local coordinates $z = (x, y)$ on a neighborhood of a fixed point (x_0, y_0) in $\mathbf{R}^p \times \mathbf{R}^q$, with $q = n - p$, so that X corresponds locally to the affine subspace $\{y = y_0\}$. Choose a local extension of the $\mathbb{G}$-plane field TX to a $\mathbb{G}$-plane field P defined on a neighborhood U of (x_0, y_0) by taking any local extension and composing it with the neighborhood retraction to $\mathbb{G}$ as in the proof of Theorem 6.2. Consider the linear operator

$$\mathbb{L}(u) \;\equiv\; \operatorname{trace}\left\{\operatorname{Hess} u\big|_{P}\right\}$$

and note that any function which is $\mathbb{G}$-psh is also $\mathbb{L}$-subharmonic on U. It will suffice to establish the linear restriction hypothesis for $\mathbb{L}$.

To see this we note that at points of X the operator $\mathbb{L}$ can be written as

$$\mathbb{L}(u) \;=\; \sum_{i,j=1}^{p} g^{ij}\left\{\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{k=1}^{p} \Gamma_{ij}^{k} \frac{\partial u}{\partial x_k}\right\} - \sum_{\alpha=1}^{q} \sum_{i,j=1}^{p} g^{ij} \Gamma_{ij}^{\alpha} \frac{\partial u}{\partial y_\alpha} \tag{7.1}$$

where g^{ij} denotes the inverse metric tensor and Γ_{ij}^{k} the Christoffel symbols of the riemannian metric in these coordinates. Equation (7.1) can be rewritten as

$$\mathbb{L}(u) \;=\; \Delta_X u - H \cdot u$$

where Δ_X is the Laplace-Beltrami operator for the induced metric on X and H is the mean curvature vector field of X. Since $H \equiv 0$ by hypothesis, the linear restriction hypothesis (Remark 5.8 $(3)'$) is satisfied and Theorem 5.7 applies to complete the proof. ∎

*where F was denoted by $\mathcal{P}^+(\mathbb{G})$.

8. $\mathbb{G}$-Flat Submanifolds.

The results of the last two sections can be expanded to a more general situation. Let Z and $\mathbb{G} \subset G(p, TZ)$ be as in Section 7. Fix a submanifold $X \subset Z$ of dimension $m \geq p$ and consider the compact subset $\mathbb{G}(TX) = \{\xi \in \mathbb{G} : \xi \subset TX\} \subset G(p, TX)$ of $\mathbb{G}$-planes tangent to X. We say that X is **$\mathbb{G}$-regular** if each tangent $\mathbb{G}$-plane at a point x can be extended to a tangent $\mathbb{G}$-plane field in a neighborhood of x in X.

The set $\mathbb{G}(TX)$ defines a subequation $F_{\mathbb{G}(TX)}$ on X by the requirement that

$$\operatorname{trace}\left\{\operatorname{Hess}_X u\big|_{\xi}\right\} \geq 0 \quad \text{for all } \xi \in \mathbb{G}(TX)$$

for C^2-functions u, where Hess_X denotes the riemannian hessian on X.

Recall that the *second fundamental form* B of X is a symmetric bilinear form on TX with values in the normal bundle NX defined by $B_{V,W} = (\nabla_V \widetilde{W})^N$ where $\widetilde{W}$ is any extension of W to a vector field tangent to X (cf. [L]) For $V, W \in T_xX$ the ambient Z-hessian and the intrinsic X-hessian differ by the second fundamental form (cf. [HL$_{2,6}$]), i.e.,

$$(\operatorname{Hess}_Z u)(V, W) \;=\; (\operatorname{Hess}_X u)(V, W) + B_{V,W}\, u \tag{8.1}$$

Definition 8.1. The submanifold X is said to be **$\mathbb{G}$-flat** if it is $\mathbb{G}$-regular and

$$\operatorname{trace}\left\{B\big|_{\xi}\right\} = 0 \quad \text{for all } \xi \in \mathbb{G}(TX).$$

THEOREM 8.2. (The Geometric Restriction Theorem). *Let $X \subset Z$ be a $\mathbb{G}$-flat submanifold. Then the restriction of any $\mathbb{G}$-plurisubharmonic function to X is $\mathbb{G}(TX)$-plurisubharmonic.*

Note. The simplest interesting case occurs when $\dim(X) = p$ and X is a $\mathbb{G}$-manifold. Then X is $\mathbb{G}$-flat if and only if it is minimal ($\mathbb{G}$-regularity holds automatically). Thus Theorem 8.2 generalizes Theorem 7.1, which in turn contains Theorem 6.2

Proof. From the $\mathbb{G}$ regularity of X we have the following elementary fact.

Lemma 8.3. *A function $u \in \mathrm{USC}(X)$ is $\mathbb{G}(TX)$psh if and only if for each tangent $\mathbb{G}$-plane field defined on an open subset $U \subset X$, the function $u\big|_U$ is $\mathbb{L}_\xi$-subharmonic, where $\mathbb{L}_\xi$ is the linear subequation on Udefined by $\mathbb{L}_\xi(v) \equiv \operatorname{tr}_\xi\{\operatorname{Hess}_X v\} \geq 0$ for $v \in C^2$.*

Proof. ($\Leftarrow$) Let φ be a test function for u at $x_0 \in X$. Fix $\xi_0 \in \mathbb{G}(T_{x_0}X)$. Extend ξ_0 to a local $\mathbb{G}(TX)$-plane field ξ. Then by assumption $\operatorname{tr}_\xi\{\operatorname{Hess}_X \varphi\} \geq 0$. This proves that $\operatorname{tr}_{\xi_0}\{\operatorname{Hess}_X \varphi\} \geq 0$ for all $\xi_0 \in \mathbb{G}(T_{x_0}X)$, i.e., $\operatorname{Hess}_{x_0}\varphi \in F_{\mathbb{G}(T_{x_0}X)}$.

($\Rightarrow$) Suppose u is $\mathbb{G}(TX)$-psh, and let ξ be a tangent $\mathbb{G}$ plane field defined on an open set $U \subset X$. Fix $x_0 \in U$ and choose a test function φ for u at x_0. Since u is $\mathbb{G}(TX)$-psh, we have $\operatorname{tr}_{\xi_0}\{\operatorname{Hess}_X \varphi\} \geq 0$ for all $\xi_0 \in \mathbb{G}(T_{x_0}X)$. Hence u is $\mathbb{L}_\xi$-subharmonic on U. ∎

The remainder of the proof of Theorem 8.2 now closely follows the argument given for the proof of Theorem 7.1, by choosing similar coordinates and extending the intrinsic operators $\mathbb{L}_\xi$ into Z. ∎

Example 8.4. ($\mathbb{G}$-regularity is necessary). Let $\mathbb{G} = \{x\text{-axis}\}$ in $\mathbf{R}^2$, and set $X = \{(x, y) : y = x^4\}$. Then X has a tangent $\mathbb{G}$-plane only at the origin. The second fundamental form (i.e., the curvature) is zero at the origin, however $\mathbb{G}$-regularity clearly fails. Restriction also fails. Consider the strictly $\mathbb{G}$-psh function $u(x, y) = \epsilon x^2 - |y|^{\frac{1}{2}}$. Then $u\big|_X = u(x, x^4) = -(1-\epsilon)x^2$ in the parameter x, and one sees easily that for ϵ small, $\mathrm{Hess}_0\, u = \frac{d^2u}{ds^2}(0) < 0$ (where $s =$ arc-length parameter).

9. Subequations on Manifolds – Automorphisms and Jet Equivalence.

In this section we suppose that a subequation F is given on a smooth manifold Z. In particular, F is a closed subset of the 2-jet bundle $J^2(Z)$. The 0-jet bundle $\mathbf{R}$ splits off as $J^2(Z) = \mathbf{R} \oplus J^2_{\text{red}}(Z)$ leaving the bundle of reduced 2-jets $J^2_{\text{red}}(Z)$. The bundle of reduced 1-jets is simply T^*Z the cotangent bundle of Z.

Restriction

If X is a submanifold of Z, let i^*_X denote the restriction of 2-jets to $X \subset Z$. Then

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Sym}^2(T^*Z) & \longrightarrow & J^2_{\text{red}}(Z) & \longrightarrow & T^*Z & \longrightarrow & 0 \\
 & & \downarrow i^*_X & & \downarrow i^*_X & & \downarrow i^*_X & & \\
0 & \longrightarrow & \mathrm{Sym}^2(T^*X) & \longrightarrow & J^2_{\text{red}}(X) & \longrightarrow & T^*X & \longrightarrow & 0
\end{array}
\tag{9.1}
$$

is commutative with exact rows. Note that $i^*_X : \mathrm{Sym}^2(T^*Z) \to \mathrm{Sym}^2(T^*Z)$ is induced by restriction of 1-forms $i^*_X : T^*Z \to T^*X$, and that it agrees with the restriction of quadratic forms on TZ to quadratic forms on TX.

Automorphisms

To begin, an automorphism of the jet bundle $J^2(Z) = \mathbf{R} \oplus J^2_{\text{red}}(Z)$ is required to split as the identity on the 0-jet factor $\mathbf{R}$ and an automorphism of the reduced jet bundle $J^2_{\text{red}}(Z)$. Hence it suffices to define automorphisms of the reduced jet bundle.

Definition 9.1. An **automorphism** of $J^2_{\text{red}}(Z)$ is a bundle isomorphism $\Phi : J^2_{\text{red}}(Z) \to J^2_{\text{red}}(Z)$ which maps the subbundle $\mathrm{Sym}^2(T^*Z)$ to itself and has the further property that this restricted isomorphism $\Phi : \mathrm{Sym}^2(T^*Z) \to \mathrm{Sym}^2(T^*Z)$ is induced by a bundle isomorphism

$$h = h_\Phi : T^*Z \longrightarrow T^*Z. \tag{9.2}$$

This means that for $A \in \mathrm{Sym}^2(T^*Z)$,

$$\Phi(A) \;=\; hAh^t, \tag{9.3}$$

that is,

$$\Phi(A)(v, w) \;=\; A(h^t v, h^t w) \qquad \text{for} \;\; v, w \in TZ.$$

Because of the upper short exact sequence in (9.1) each automorphism Φ of $J^2_{\rm red}(Z)$ induces a bundle isomorphism

$$g = g_\Phi : T^*Z \ \to \ T^*Z. \tag{9.4}$$

This bundle isomorphism is *not* required to agree with h in (9.2).

Lemma 9.2. *The automorphisms of $J^2(Z)$ form a group. They are the sections of the bundle of groups whose fibre at $z \in Z$ is the group of automorphisms of $J^2_z(Z)$ defined above.*

Proof. See [HL$_6$, §4].

Proposition 9.3. *With respect to any splitting*

$$J^2(Z) \ = \ \mathbf{R} \oplus T^*Z \oplus {\rm Sym}^2(T^*Z)$$

of the upper short exact sequence (9.1), a bundle automorphism has the form

$$\Phi(r,p,A) \ = \ (r, gp, hAh^t + L(p)) \tag{9.5}$$

where g and h are smooth sections of the bundle ${\rm End}(T^*Z)$ *and L is a smooth section of the bundle* ${\rm Hom}\,(T^*Z, {\rm Sym}^2(T^*Z))$.

Proof. Obvious.

Example 1. The trivial 2-jet bundle on $\mathbf{R}^n$ has fibre

$$\mathbf{J}^2 = \mathbf{R} \times \mathbf{R}^n \times {\rm Sym}^2(\mathbf{R}^n).$$

with automorphism group

$${\rm Aut}(\mathbf{J}^2) \ \equiv \ {\rm GL}_n \times {\rm GL}_n \times {\rm Hom}\,(\mathbf{R}^n, {\rm Sym}^2(\mathbf{R}^n))$$

where the action is given by

$$\Phi_{(g,h,L)}(r,p,A) \ = \ (r, \ gp, \ hAh^t + L(p)).$$

and the group law is

$$(\bar g, \bar h, \bar L)\cdot(g,h,L) \ = \ (\bar g g, \ \bar h h, \bar h L \bar h^t + \overline{L} \circ g)$$

Example 2. Given a local coordinate system $(x_1, ..., x_n)$ on an open set $U \subset Z$, the canonical trivialization

$$J^2(U) \ = \ U \times \mathbf{R} \times \mathbf{R}^n \times {\rm Sym}^2(\mathbf{R}^n) \tag{9.6}$$

is determined by the coordinate 2-jet $J_x u = (u, Du, D^2u)$ evaluated at x. With respect to this splitting, every automorphism is of the form

$$\Phi(u, Du, D^2u) \ = \ (u,\ gDu,\ h \cdot D^2u \cdot h^t + L(Du)) \tag{9.7}$$

where $g_x, h_x \in \mathrm{GL}_n$ and $L_x : \mathbf{R}^n \to \mathrm{Sym}^2(\mathbf{R}^n)$ is linear for each point $x \in U$.

Jet Equivalence

Definition 9.4. Two subequations $F, F' \subset J^2(Z)$ are **jet equivalent** if there exists an automorphism $\Phi : J^2(Z) \to J^2(Z)$ with $\Phi(F) = F'$.

Definition 9.5. A subequation $F \subset J^2(Z)$ is **locally jet equivalent to a constant coefficient subequation** if each point x has a distinguished coordinate neighborhood U so that $F\big|_U$ is equivalent to a constant coefficient subequation $U \times \mathbf{F}$ in those distinguished coordinates.

Lemma 9.6. *Suppose Z is connected and $F \subset J^2(Z)$ is locally jet equivalent to a constant coefficient subequation. Then there is a subequation $\mathbf{F} \subset \mathbf{J}^2$, unique up to equivalence, such that F is locally jet equivalent to $U \times \mathbf{F}$ on every distinguished coordinate chart.*

Proof. In the overlap of any two distinguished charts $U_1 \cap U_2$ choose a point x. Then the local equivalences Φ_1 and Φ_2, restricted to F_x, determine an equivalence from $\mathbf{F}_1$ to $\mathbf{F}_2$. Thus the local constant coefficient equations on these charts are all equivalent, and they can be made equal by applying the appropriate constant equivalence on each chart. ■

Remark 9.7. One reason that the notion of equivalence employed in Definition 9.3 is natural is that this notion is induced by diffeomorphisms. Namely, if φ is a diffeomorphism fixing a point x_0, then in local coordinates (as in Example 2 above) the right action on $J^2_{x_0}$, induced by the pull-back φ^* on 2-jets, is given by (9.7) where h_{x_0} is the transpose on the Jacobian matrix $((\frac{\partial \varphi^i}{\partial x_j}))$ and $L_{x_0}(Du) = \sum_{k=1}^n u_k \frac{\partial^2 \varphi^k}{\partial x_i \partial x_j}(x_0)$.

Cautionary Note. A local equivalence $\Phi : F \to F'$ does not take F-subharmonic functions to F'-subharmonic functions. In fact, for $u \in C^2$, $\Phi(J^2u)$ is almost never the 2-jet of a function. It happens if and only if $\Phi(J^2u) = J^2u$.

Relative Automorphisms and Relative Jet Equivalence

Suppose now that $i : X \hookrightarrow Z$ is an embedded submanifold.

Definition 9.8. A **relative automorphism of $J^2(Z)$ with respect to X** is an automorphism $\Phi : J^2(Z) \to J^2(Z)$ such that on X the diagram

$$\begin{array}{ccc} J^2(Z) & \xrightarrow{\Phi} & J^2(Z) \\ i^* \downarrow & & \downarrow i^* \\ J^2(X) & \xrightarrow{\varphi} & J^2(X) \end{array}$$

commutes for some automorphsim $\varphi : J^2(X) \to J^2(X)$.

Relative automorphisms with respect to X are a subgroup of the automorphisms of $J^2(Z)$.

Fix a splitting $J^2(Z) = \mathbf{R} \oplus T^*Z \oplus \mathrm{Sym}^2(T^*Z)$, and let g, h and L be associated to an automorphism Φ as in Proposition 9.4. Then one easily checks that: *Φ is a relative automorphism of $J^2(Z)$ with respect to X if and only if*

$$g^t(TX) \subset TX, \qquad h^t(TX) \subset TX \qquad \text{and} \qquad L_{N^*X,\mathrm{Sym}^2(T^*X)} = 0. \tag{9.8}$$

Here $L_{N^*X,\mathrm{Sym}^2(T^*X)}$ denotes the restriction of L to N^*X followed by the restriction of quadratic forms in $\mathrm{Sym}^2(T^*Z)$ to $\mathrm{Sym}^2(T^*X)$.

Definition 9.9. Two subequations $F, F' \subset J^2(Z)$ are **jet equivalent modulo** X if $F' = \Phi(F)$ for some relative automorphism Φ with respect to X.

If $F, F' \subset J^2(Z)$ are jet equivalent modulo X, then the induced subequations $H = i^*F$ and $H' = i^*F'$ are jet equivalent on X.

By an *adapted coordinate neighborhood* of a point $z_0 = (x_0, y_0) \in X$ we mean a local coordinate system $z = (x, y)$ on a neighborhood U of z_0 such that $X \cap U = \{(x, y) : y = y_0\}$.

Definition 9.10. The subequation $F \subset J^2(Z)$ is **locally jet equivalent modulo X to a constant coefficient subequation** if each point in X has an adapted coordinate neighborhood U so that $F\big|_U$ is jet equivalent modulo X to a constant coefficient subequation $U \times \mathbf{F}$ in those adapted coordinates.

Now we examine what this means in more detail. Suppose that $z = (x, y) \in \mathbf{R}^N = \mathbf{R}^n \times \mathbf{R}^m$ is the adapted coordinate system and $\Phi : J^2(U) \to J^2(U)$ is the jet equivalence modulo X. By Proposition 9.3, Φ acting on a coordinate 2-jet (u, Du, D^2u) must be of the form

$$\Phi(J) = \Phi(u, Du, D^2u) = (u, gDu, hD^2uh^t + L(Du)). \tag{9.8}$$

Moreover, we have

$$J \in F \quad \Longleftrightarrow \quad \Phi(J) \in \mathbf{F}. \tag{9.9}$$

With respect to the splitting $\mathbf{R}^n \times \mathbf{R}^m$ into x and y coordinates, each coordinate 2-jet J can be written as

$$J = \left(r, (p, q), \begin{pmatrix} A & C \\ C^t & B \end{pmatrix}\right), \qquad \text{and} \qquad i^*(J) = (r, p, A).$$

is the restriction of J to X. The sections g and h can be written in block form as

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \qquad \text{and} \qquad h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}. \tag{9.10}$$

Also L can be decomposed into the sum $L = L' + L''$ where $L' \in \mathrm{End}(\mathbf{R}^n, \mathrm{Sym}^2(\mathbf{R}^N))$ and $L'' \in \mathrm{End}(\mathbf{R}^m, \mathrm{Sym}^2(\mathbf{R}^N))$. Each of L, L', L'' can be blocked into $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$ components in $\mathrm{Sym}^2(\mathbf{R}^n \oplus \mathbf{R}^m)$, in analogy with g and h above.

Now we can compute the restriction $i^*\Phi(J)$ of $\Phi(J)$. Namely,

$$
\begin{aligned}
i^*\Phi(J) \;=\; (r,\; g_{11}p + g_{12}q,\;\; & h_{11}Ah_{11}^t + h_{12}C^t h_{11}^t \\
& + h_{11}Ch_{12}^t + h_{12}Bh_{12}^t + L_{11}'(p) + L_{11}''(q))
\end{aligned}
\tag{9.11}
$$

In order for Φ to be a jet equivalence modulo X this must agree with an automorphism $\varphi : J^2(U\cap X) \to J^2(U\cap X)$, which is the case if and only if on X

$$
g_{12} \;=\; 0, \quad h_{12} \;=\; 0, \quad \text{and} \quad L_{11}'' \;=\; 0 \tag{9.12}
$$

so that

$$
\varphi(r,p,A) \;=\; \big(r,\; g_{11}p,\; h_{11}Ah_{11}^t + L_{11}'(p)\big) \tag{9.13}
$$

10. A Second Restriction Theorem.

Although the next result does not include the Geometric Restriction Theorem, it does apply to some interesting non-geometric cases.

THEOREM 10.1. *Let $i : X \hookrightarrow Z$ be an embedded submanifold and $F \subset J^2(Z)$ a subequation. Assume that F is locally jet equivalent modulo X to a constant coefficient subequation $\mathbf{F}$. If $\mathbf{H} \equiv i^*\mathbf{F}$ is a closed set, then $H \equiv i_X^* F$ is locally jet equivalent to the constant coefficient subequation $\mathbf{H}$, and restriction holds. That is,*

$$
u \;\text{ is } F \text{ subharmonic on }\; Z \qquad \Rightarrow \qquad u\big|_X \;\text{ is } H \text{ subharmonic on }\; X
$$

Proof. Adopt the notation following Definition 9.10. By hypothesis (9.12) we have that

$$
g_{12}(x,y) \;\text{ and }\; h_{12}(x,y) \text{ are } O(|y-y_0|) \qquad \text{and} \qquad L''{}_{11}(x,y) \;=\; O(|y-y_0|). \tag{10.1}
$$

We now show that F satisfies the Restriction Hypothesis. Fix $(r_0,p_0,A_0) \in J^2_{x_0}(X)$ and suppose there are sequences $z_\epsilon = (x_\epsilon, y_\epsilon)$ and r_ϵ with

$$
J_\epsilon \;=\; \left(r_\epsilon, \left(p_0 + A_0(x_\epsilon - x_0), \frac{y_\epsilon - y_0}{\epsilon}\right), \begin{pmatrix} A_0 & 0 \\ 0 & \frac{1}{\epsilon} I \end{pmatrix}\right) \;\in\; F_{z_\epsilon} \tag{10.2}
$$

and

$$
x_\epsilon \;\to\; x_0, \quad \frac{|y_\epsilon - y_0|^2}{\epsilon} \;\to\; 0, \quad r_\epsilon \;\to\; r_0,
$$

as $\epsilon \to 0$. Now (10.2) is equivalent to the fact that

$$
\Phi_{z_\epsilon}(J_\epsilon) \;\in\; \mathbf{F} \;\; \text{for all } \epsilon. \tag{10.3}
$$

This means that the $(1,1)$-component

$$
i^*\Phi_{z_\epsilon}(J_\epsilon) \;\in\; i^*\mathbf{F} \;\equiv\; \mathbf{H} \;\; \text{for all } \epsilon. \tag{10.4}
$$

It will suffice to show that

$$i^*\Phi_{z_\epsilon}(J_\epsilon) \text{ converges to } \varphi(r_0, p_0, A_0) \text{ as } \epsilon \to 0. \tag{10.5}$$

Write

$$i^*\Phi_{z_\epsilon}(J_\epsilon) \;=\; (r_\epsilon, p_\epsilon, A_\epsilon)\,.$$

By (9.11)

$$p_\epsilon \;=\; g_{12}(z_\epsilon)(p_0 + A_0(x_\epsilon - x_0) + g_{12}\tfrac{1}{\epsilon}(y_\epsilon - y_0).$$

Now (10.1) implies that $p_\epsilon \to g_{11}(z_0)p_0$. Furthermore, by (9.11)

$$\begin{aligned} A_\epsilon \;=\; & h_{11}(z_\epsilon)A_0 h_{11}^t(z_\epsilon) + \tfrac{1}{\epsilon}h_{12}(z_\epsilon)h_{12}^t(z_\epsilon) \\ & + L'_{11}\cdot(p_0 + A_0(x_\epsilon - x_0)) + L''_{11}\cdot((\tfrac{1}{\epsilon}(y_\epsilon - y_0)). \end{aligned}$$

Again by (10.1) we have $A_\epsilon \to h_{11}(z_0)A_0h_{11}^t(z_0) + L'_{11}(p_0)$. Since $\varphi_{z_0}(r_0,p_0,A_0) = (r_0,\ g_{11}(z_0)p_0,\ h_{11}(z_0)A_0h_{11}^t(z_0) + L'_{11}(p_0))$, this completes the proof. ∎

11. Applications of the Second Restriction Theorem.

The Second Restriction Theorem has a number of interesting applications. One is to prove restriction for the intrinsically defined plurisubharmonic functions on an almost complex manifold (see [HL$_8$]). Another, which we present below, applies to universally defined subequations on manifolds with topological G-structure (as in [HL$_6$]).

We begin with the case of universal riemannian subequations. Let

$$\mathbf{F} \;\subset\; \mathbf{J}^2_N = \mathbf{R}\times\mathbf{R}^N\times \mathrm{Sym}^2(\mathbf{R}^N) \tag{11.1}$$

be a closed subset with the properties that:

(1) $\mathbf{F} + (\mathbf{R}_- \times\{0\}\times\mathcal{P}) \subset \mathbf{F}$, where $\mathcal{P} \equiv \{A\in\mathrm{Sym}^2(\mathbf{R}^N) : A \geq 0\}$,

(2) $\mathbf{F} = \overline{\mathrm{Int}\mathbf{F}}$, and

(3) $\mathbf{F}$ is invariant under the natural action of O_N on $\mathbf{J}^2_N$.

Let Z be a riemannian manifold of dimension N and recall the canonical splitting

$$J^(Z) \;=\; \mathbf{R}\times T^*Z\times\mathrm{Sym}^2(T^*Z) \tag{11.2}$$

given by the riemannian hessian

$$(\mathrm{Hess}\, u)(V,W) \;\equiv\; VWu - (\nabla_V W)u \tag{11.3}$$

(for vector fields V and W; see [HL$_6$].)

Definition 11.1. The universal subequation $\mathbf{F}$ in (11.1) canonically determines a subequation $F\subset J^2(Z)$ on any riemannian N-manifold Z by the requirement that

$$Ju_z = (u(z), (du)_z, \mathrm{Hess}_z u) \;\in\; F_z \qquad\Longleftrightarrow\qquad [u(z), (du)_z, \mathrm{Hess}_z u] \;\in\; \mathbf{F} \tag{11.4}$$

where $[u(z), (du)_z, \mathrm{Hess}_z u]$ denotes the coordinate representation of $(u(z), (du)_z, \mathrm{Hess}_z u)$ with respect to any orthonormal basis of $T_z Z$. We call F **the subequation on Z canonically determined by $\mathbf{F}$**.

THEOREM 11.2. *Let Z be a riemannian manifold of dimension N and $F \subset J^2(Z)$ a subequation canonically determined by an O_N-invariant universal subequation $\mathbf{F} \subset \mathbf{J}^2_N$ as above. Then restriction holds for F to any totally geodesic submanifold $X \subset Z$.*

Proof. The theorem is local, so we may restrict to the case where

$$\begin{aligned} Z &\equiv \{x = (x', x'') \in \mathbf{R}^n \times \mathbf{R}^m : |x'| < 1, |x''| < 1\}, \quad \text{and} \\ X &\equiv \{x = (x', 0) \in \mathbf{R}^n \times \mathbf{R}^m : |x'| < 1\}, \end{aligned}$$

with $n + m = N$. We may furthermore assume that

$$\partial'_i \ \perp\ \partial''_j \quad \text{along } X \quad \text{for all } i, j \tag{11.5}$$

in the given metric on Z where

$$\partial'_i \equiv \frac{\partial}{\partial x'_i} \qquad \text{and} \qquad \partial''_j \equiv \frac{\partial}{\partial x''_j}.$$

To see this we choose our coordinates as follows. First choose a local coordinate map $\varphi : \{x', |x'| \le 1\} \to X$. Fix a basis $\nu_1, ..., \nu_m$ of the normal space to X at $\varphi(0)$ and extend them to normal vector fields $\nu_1, ..., \nu_m$ on X by parallel translation along the curves corresponding to rays from the origin in the disk $\{x', |x'| \le 1\}$. Applying the exponential map to $x''_1 \nu_1 \varphi((x')) + \cdots + x''_m \nu_m(\varphi(x'))$ gives the desired coordinates for $|x''| <$ some ϵ. (Of course, one can then renormalize to $|x''| < 1$.)

We now choose an orthonormal frame field $(e_1, ..., e_{n+m}) = (e'_1, ..., e'_n, e''_1, ..., e''_m)$ on Z (with respect to the given metric) so that along X

$$e'_1, ..., e'_n \ \text{ are tangent to } \ X \qquad \text{and} \qquad e''_1, ..., e''_m \ \text{ are normal to } \ X. \tag{11.6}$$

Our subequation $F \subset J^2(Z)$ is then given explicitly by the condition

$$\big(u, (e_1 u, ..., e_{n+m} u), \mathrm{Hess}\, u(e_i, e_j)\big)_z \ \in\ \mathbf{F} \tag{11.7}$$

for $z \in Z$. We now write

$$e_i \ = \ \sum_{j=1}^{n+m} h_{ij} \partial_j \qquad \text{for} \quad i = 1, ..., n+m$$

where $\partial \equiv (\partial', \partial'')$. From (11.5) we have that the matrix h decomposes as

$$h \ = \ \begin{pmatrix} h' & 0 \\ 0 & h'' \end{pmatrix} \qquad \text{along } \ X. \tag{11.8}$$

We now compute that

$$e_i u \;=\; \sum_j h_{ij}\partial_j u, \qquad \text{and}$$

$$\begin{aligned}
(\mathrm{Hess}\, u)(e_i, e_j) \;&=\; (\mathrm{Hess}\, u)\left(\sum_k h_{ik}\partial_k, \sum_\ell h_{j\ell}\partial_\ell\right) \;=\; \sum_{k,\ell} h_{ik}h_{j\ell}(\mathrm{Hess}\, u)(\partial_k, \partial_\ell) \\
&=\; \sum_{k,\ell} h_{ik}h_{j\ell}\left\{\partial_k\partial_\ell u - \left(\nabla_{\partial_k}\partial_\ell\right) u\right\} \\
&=\; \sum_{k,\ell} h_{ik}h_{j\ell}\left\{\partial_k\partial_\ell u - \sum_m \Gamma_{k\ell}^m \partial_m u\right\}
\end{aligned} \qquad .$$

where $\Gamma = \{\Gamma_{k\ell}^m\}$ are the classical Christoffel symbols. Expressed briefly, we have that

$$e \cdot u \;=\; hDu \qquad \text{and} \qquad (\mathrm{Hess}\, u)(e_*, e_*) \;=\; h(D^2u)h^t - \widetilde{\Gamma} \cdot Du$$

where $\widetilde{\Gamma} \equiv h\Gamma h^t$. Thus our condition (11.7) can be rewritten in terms of the coordinate jets as

$$\left(u,\, hDu,\, h(D^2u)h^t - \widetilde{\Gamma} \cdot Du\right) \;\in\; \mathbf{F}. \tag{11.9}$$

This says precisely that our subequation F is jet equivalent to the constant coefficient subequation $\mathbf{F}$ in these coordinates.

We claim that this is an equivalence mod X. For this we must establish the conditions in (9.12). Note first that in this case $g = h$ and $h_{12} = 0$ by (11.8). For the last condition we use the fact that X is totally geodesic. This means precisely that

$$\nabla_{\partial_i'}\partial_j' \;=\; \sum_{k=1}^n \Gamma_{ij}^k \partial_k' \qquad \text{along} \;\; X,$$

i.e. $\nabla_{\partial_i'}\partial_j'$ has no normal components along X for all $1 \leq i, j \leq n$. This is exactly the third condition in (9.12).

Theorem 11.2 now follows from Theorem 10.1. ∎

Theorem 11.2 can be extended to the case where the riemannian manifold Z has a topological reduction of the structure group to a subgroup

$$G \;\subset\; \mathrm{O}_N.$$

Such a reduction consists of an open covering $\{U_\alpha\}_\alpha$ of Z and an orthonormal tangent frame field $e^\alpha = (e_1^\alpha, ..., e_N^\alpha)$ given on each open set U_α with the property that the change of framings

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \;\longrightarrow\; G \;\subset\; \mathrm{O}_N$$

take their values in G.

The local frame fields e_a are called **admissible**. Note that if e on U is an admissible frame field, one can add to the family of admissible framings, any frame field of the form

ge where $g : U \to G$ is a smooth map. We assume that our G-structure has a maximal family of admissible frame fields.

Definition 11.3. Suppose Z has a topological G-structure. A submanifold $X \subset Z$ is a **G-submanifold** if for every point $z \in X$ there is an admissible framing e on a neighborhood U of z such that on $X \cap U$

$$e_1, ..., e_n \ \text{ are tangent to } \ X \cap U \qquad \text{and} \qquad e_{n+1}, ..., e_N \ \text{ are normal to } \ X \cap U. \tag{11.10}$$

Example 11.4. Suppose $G = \mathrm{U}_m \subset \mathrm{O}_{2m}$. Having a U_m-structure on Z is equivalent to having an orthogonal almost complex structure $J : TZ \to TZ$, $J^2 \equiv -I$ on Z. A U_m-submanifold $X \subset Z$ is simply an almost complex submanifold, i.e., having the property that $J(T_xX) = T_xX$ for all $x \in X$.

On a manifold with topological G-structure, we can enlarge the set of universal subequations by replacing property (3) above with

(3)′ $\mathbf{F}$ is invariant under the natural restricted action of G on $\mathbf{J}^2_N$.

As above any such set $\mathbf{F}$ determines a subequation F on Z.

THEOREM 11.5. *Let Z be a riemannian manifold with topological G-structure, and $F \subset J^2(Z)$ a subequation canonically determined by a G-invariant universal subequation $\mathbf{F} \subset \mathbf{J}^2_N$ satisfying (1), (2) and (3)′. Then restriction holds for F to any totally geodesic G-submanifold $X \subset Z$.*

Proof. The proof exactly follows the one given for Theorem 11.2. One merely has to choose the local frame field e with property (11.6) to be an admissible field (cf. (11.10)). Details are left to the interested reader. ■

Note 11.6. Every almost complex manifold (Z, J) admits many almost complex submanifolds of dimension one (pseudo-holomorphic curves) by a classical result of Nijenhuis and Woolf [NW]. In fact there exist pseudo-holomorphic curves in every complex tangent direction at every point, and these curves can be used to define plurisubharmonic functions and to give a complex Monge-Ampère operator (see [HL$_8$]). However, examples show that this operator does not in general agree with the one given above by a choice of hermitian metric (i.e., a topological U_m-structure) on Z. However, the Second Restriction Theorem is also crucial in the study of this intrinsically defined potential theory [HL$_8$].

Example 11.7. Theorem 11.2 asserts that every universal riemannian subequation satisfies restriction to totally geodesic submanifolds. Of course for some subequations this restriction is trivial, i.e., $i^*F = J^2(X)$. One example of this is the Laplace-Beltrami equation given by $\mathbf{F} = \{(r, p, A) : \mathrm{tr}A \geq 0\}$. Nevertheless, there are also many subequations which have interesting restrictions. One such is the classical $\mathbf{F} = \{(r, p, A) : A \geq 0\}$ corresponding to riemannian convex functions. This also falls under the aegis of Theorem 8.2. A case which is not covered by previous results is the following.

Given $A \in \mathrm{Sym}^2(\mathbf{R}^N)$, let $\lambda_1(A) \leq \cdots \leq \lambda_N(A)$ denote the *ordered eigenvalues* of A, and define for $\mu \in \mathbf{R}$

$$\mathbf{F}_{k,\mu} \ \equiv \ \{(r, p, A) \in \mathbf{J}^2_N : \lambda_k(A) \geq \mu\}.$$

Let $F_{k,\mu}(Z)$ be the induced subequation. Then one computes that for a submanifold $i : X \subset Z$

$$i^* F_{k,\mu}(Z) \;=\; F_{k,\mu}(X). \tag{11.11}$$

This follows from the fact that

$$\lambda_k(A) \;=\; \sup_V \left\{ \inf_{v \in V'} \frac{\langle Av, v\rangle}{\|v\|^2} \right\}$$

where the supremum is taken over subspaces $V \subset \mathbf{R}^N$ of codimension $k-1$.

Example 11.8. One can use Theorem 10.2 to treat inhomogeneous subequations with variable RHS. For example the subequation

$$\lambda_k(A) \;\geq\; f(z)$$

where f is any smooth positive function on Z, satisfies restriction to all totally geodesic submanifolds. To see this, start with the universal riemannian subequation $F_{k,1}$ corresponding to $\lambda_k(A) \geq 1$ discussed above, and apply the local jet-equivalence $\Phi : F_{k,1} \to F_{k,1}$ given by $\Phi(r, pA) = (r, p, (hI)A(hI)^t) = (r, p, h^2 A)$ where $h^{-2} = f$.

Appendix A. Elementary Examples Where Restriction Fails.

As noted in Examples 5.2 and 8.4 restriction may fail. Here are two more elementary examples where restriction, and therefore also the Restriction Hypothesis, fail. .

Example A.1. (First Order). Define F on $\mathbf{R}^2$ by $p \pm y^i q^j \geq 0$ (where i and j are positive integers).

Case $j > i$. Restriction to $\{y = 0\}$, and hence the restriction hypothesis, fails. Consider $u(x, y) = -x + \frac{1}{\alpha}|y|^\alpha$ with $\alpha > 0$ small. Then $p = -1$, and with the right choice of $\pm$ we have $\pm y^i q^j = |y|^{i+j\alpha-j}$. Thus $p \pm y^i q^j = -1 + |y|^\beta \geq 0$ with $\beta < 0$. This proves that u is F-subharmonic if $|y| > 0$ is small. At points $(x, y) = (x, 0)$ there are no test functions. Thus u is F-subharmonic. However, the restriction $u\big|_X = -x$ is not $H \equiv i^* F$-subharmonic, since H is defined by $p \geq 0$.

Case $j \leq i$. The restriction hypothesis, and hence restriction, holds on $\{y = 0\}$. Assume (4.5 and 6). Define $p_\epsilon \equiv p_0 + A_0(x_\epsilon - x_0)$ and $q_\epsilon \equiv \frac{1}{\epsilon}(y_\epsilon - y_0) = \frac{1}{\epsilon} y_\epsilon$. By (4.5) we know that $p_\epsilon \pm y_\epsilon^i q_\epsilon^j \geq 0$. By (4.6) we have that $p_\epsilon \to p_0$ and $|y_\epsilon^i q_\epsilon^j| = \frac{1}{\epsilon^j}|y_\epsilon^{i+j}| \leq |\frac{y_\epsilon^2}{\epsilon}|^j \to 0$. This proves $p_0 \geq 0$. ■

Example A.2. (Second Order). Let $Z = \mathbf{R}^n \times \mathbf{R}^m$ with coordinates (x, y) and set $X = \{y = 0\}$. Consider the linear subequation F defined in terms of the notation (4.1) by

$$\mathrm{tr} A + |y|^\beta \mathrm{tr} B \;\geq\; 0.$$

for a constant $\beta > 0$. Fix ϵ, assume $\beta < \epsilon < 2$ and define

$$u(x,y) \ = \ -\frac{1}{2}|x|^2 + \frac{1}{2-\epsilon}|y|^{2-\epsilon}.$$

This function is F-subharmonic for $|y|$ small. To see this first note that $\mathrm{Hess}(\frac{1}{2-\epsilon}|y|^{2-\epsilon}) = |y|^{-\epsilon}\{I - \epsilon\hat{y}\circ\hat{y}\}$ where $\hat{y} = y/|y|$. Hence $\mathrm{tr}\{\mathrm{Hess}(\frac{1}{2-\epsilon}|y|^{2-\epsilon})\} = (m-\epsilon)|y|^{-\epsilon}$. For $y \neq 0$ the F-condition on u is that $-n + (m-\epsilon)|y|^{\beta-\epsilon} \geq 0$. Since $\beta - \epsilon < 0$, this holds for all $|y| > 0$ sufficiently small. At points of $y = 0$ one sees that there are no test functions for u. Hence, u is F-subharmonic for $|y|$ small, as claimed.

Observe now that the restricted subequation H on $\{y = 0\}$ is just $\Delta_x u \geq 0$, which fails in this case. Hence, restriction and therefore also the restriction hypothesis fail in this case.

Appendix B. Restriction of Sets of Quadratic Forms Satisfying Positivity.

In this Appendix we provide the basic linear algebra material used in our restriction theorems and their applications.

Let $\mathrm{Sym}^2(T^*)$ denote the vector space of quadratic forms on a finite dimensional real vector space T. Let $\mathcal{P} \subset \mathrm{Sym}^2(T^*)$ denote the subset of non-negative quadratic forms. A subset $F \subset \mathrm{Sym}^2(T^*)$ is said to satisfy **positivity (P)** if

$$F + \mathcal{P} \ \subset \ F. \qquad (B.1)$$

Lemma B.1. *If $F \subset \mathrm{Sym}^2(T^*)$ is a closed set satisfying positivity, then*

(a) $\quad F + \mathrm{Int}\mathcal{P} \ \subset \ \mathrm{Int}F,$

(b) $\quad F = \overline{\mathrm{Int}F},$

(c) $\quad \mathrm{Int}F + \mathcal{P} \ \subset \ \mathrm{Int}F.$

If, in addition, F is a cone with vertex at the origin, then

(d) $\quad F = \mathrm{Sym}^2(T^*) \quad \iff \quad \exists\, A \in F$ *with* $A < 0$.

Proof. (a) Note that $A + \mathrm{Int}\mathcal{P}$ is an open subset of F for each $A \in F$.

(b) Pick $P \in \mathrm{Int}\mathcal{P}$, i.e., $P > 0$. Then by (a) we have that $A \in F \ \Rightarrow\ A + \epsilon P \in \mathrm{Int}F$ for each $\epsilon > 0$.

(c) Note that $\mathrm{Int}F + P$ is an open subset of F for each $P \in \mathcal{P}$.

(d) Suppose F contains a negative definite $A < 0$. Then for each $B \in \mathrm{Sym}^2(T^*)$, if $t >> 0$ is large enough, $P \equiv B - tA$ is positive. Hence, $B = tA + P \in tF + \mathcal{P} \subset F$. ∎

Given a subspace $W \subset T$, let $i_W^* A = A\big|_W$ denote the restriction of a quadratic form $A \in \mathrm{Sym}^2(T^*)$ to W. Thus $i_W^* : \mathrm{Sym}^2(T^*) \to \mathrm{Sym}^2(W^*)$.

THEOREM B.2. *Suppose that F is a closed subset of $\mathrm{Sym}^2(T^*)$ which is both a cone and satisfies (P). The following conditions on a proper subspace $W \subset T$ are equivalent.*

*(1) (W **is F-Morse**) There exists $A \in F$ with $i_W^* A < 0$.*

(2) (F **is unconstrained by** *W) $i_W^* F = \mathrm{Sym}^2(W^*)$ or equivalently $F + \ker i_W^* = \mathrm{Sym}^2(T^*)$.*

(3) Given $B \in \ker i_W^*$, if $B \geq 0$ and rank $B = \mathrm{codim} W$, then$B \in \mathrm{Int} F$.

(3)$'$ (W **has an F-strict complement***) There exists $B \in \mathrm{Int} F$ with $i_W^* B = 0$.*

Remark B.4. If T has an inner product and F is geometrically defined by $\mathbb{G} \subset G(p,T)$, then these conditions are equivalent to the condition that W contains no $\mathbb{G}$-planes. (See Corollary B.9.) This justifies the following terminology.

Definition B.5. A subspace W satisfying the conditions in Theorem B.2 will be called **totally F-free**.

Proof. Conditions (1) and (2) are equivalent by (d) above. Obviously (3) $\Rightarrow$ (3)$'$ since $B \geq 0$ with $i_W^* B = 0$ and $\mathrm{rank}\, B = \mathrm{codim} W$ always exist.

Next we prove that (3)$'$ $\Rightarrow$ (1). If $B \in \mathrm{Int} F$, then $A \equiv B - \epsilon P \in F$ with $P > 0$ and $\epsilon > 0$ small. If $i_W^* B = 0$, then $i_W^* A = -\epsilon i_W^* P < 0$ since the restriction of a positive definite quadratic form is also positive definite.

Finally we show that (1) $\Rightarrow$ (3). Choose $A \in F$ with $i_W^* A < 0$. Suppose that B satisfies the hypothesis of (3). Pick N transverse to W with $T = W \oplus N$. Then in block form

$$A \equiv \begin{pmatrix} -a & c \\ c^t & b \end{pmatrix} \qquad \text{and} \qquad B \equiv \begin{pmatrix} 0 & \gamma \\ \gamma^t & \beta \end{pmatrix}$$

where $a = -i_W^* A > 0$ and $0 = i_W^* B$. Since $B \geq 0$, it is a standard fact that $\gamma = 0$. Since $\mathrm{rank}\, B = \dim N$, we must have $\beta > 0$. Set

$$P \equiv \tfrac{1}{t} B - A = \begin{pmatrix} a & -c \\ -c^t & \frac{1}{t}\beta - b \end{pmatrix}.$$

Since $a, \beta > 0$, one can show that $P > 0$ if $t > 0$ is sufficiently small. Hence, $B = tA + tP \in F + \mathrm{Int}\mathcal{P} \subset \mathrm{Int} F$ since F is a cone satisfying positivity. ∎

Using this algebra one can prove the following topological result which is a vast generalization of a theorem of Andreotti-Frankel. Given a subequation F on a domain Ω we define the *free dimension* $\dim_{\mathrm{fr}}(F)$ of F to be the largest dimension of a tangent subspace $W \subset T\Omega$ which is F-free. We say F is *conical* if each F_x is a cone with vertex at the origin.

THEOREM B.6. *Let F be a conical subequation on a domain Ω in a manifold Z. If Ω admits a strictly F-subharmonic exhaustion function (i.e., if Ω is* **strictly F-convex***), then Ω has the homotopy-type of a CW-complex of dimension $\leq \dim_{\mathrm{fr}}(F)$.*

Proof. This follows from Morse theory and Theorem B.2 (1) above applied to the Hessian of the exahustion function at its critical points (cf. [HL$_4$]). ∎

Geometrically Determined Subsets of $\mathrm{Sym}^2(T^*)$

Now we assume that T is an inner product space. Then the **trace** of $A \in \mathrm{Sym}^2(T^*)$ is well defined, and induces an inner product $\langle A, B\rangle = \mathrm{trace}(AB)$ on $\mathrm{Sym}^2(T^*)$. Let $G(p,T)$ denote the grassmannian of p-planes in T. By identifying a subspace a subspace $V \subset T$

with orthogonal projection P_V onto V we can consider the grassmannian $G(p,T)$ to be a subset of $\mathrm{Sym}^2(T^*)$. The **V-trace** of $A \in \mathrm{Sym}^2(T^*)$ is defined by

$$\mathrm{tr}_V A \;=\; \mathrm{trace}\,(i_V^* A) \;=\; \langle P_V, A\rangle.$$

Definition B.7. Given a closed subset $\mathbb{G}$ of the grassmannian, the subset $F_{\mathbb{G}} \subset \mathrm{Sym}^2(T^*)$ defined by

$$A \in F_{\mathbb{G}} \qquad \Longleftrightarrow \qquad \mathrm{tr}_V A \geq 0 \;\; \forall\, V \in \mathbb{G} \tag{B.2}$$

is said to be **geometrically determined by $\mathbb{G}$**.

Definition B.8. Given a closed subset $\mathbb{G} \subset \Gamma(p,T)$ and a subspace $W \subset T$ of dimension $\geq p$, the W **-tangential part of $\mathbb{G}$** is defined to be

$$\mathbb{G}(W) \;\equiv\; \{V \in \mathbb{G} : V \subset W\} \tag{B.3}$$

and we say that $V \in \mathbb{G}(W)$ is **tangential to** W.

THEOREM B.9. *Suppose that $F_{\mathbb{G}}$ is geometrically determined by the closed subset $\mathbb{G} \subset G(p,T)$. Then for each subspace $W \subset T$ the restriction of $F_{\mathbb{G}}$ to W is geometrically determined by the tangential part of $\mathbb{G}$. That is*

$$i_W^* F_{\mathbb{G}} \;=\; F_{\mathbb{G}(W)}. \tag{B.4}$$

Definition B.10. The subspace W is **totally $\mathbb{G}$-free** if the tangential part of $\mathbb{G}$ is empty (i.e., $\mathbb{G}(W) = \emptyset$)or equivalently $F_{\mathbb{G}(W)} = \mathrm{Sym}^2(W^*)$.

Corollary B.11.

$$\begin{aligned} & i_W^* F_{\mathbb{G}} = \mathrm{Sym}^2(W^*) && \textit{(i.e., W is totally $F_{\mathbb{G}}$-free)} \\ \Longleftrightarrow \quad & \mathbb{G}(W) = \emptyset && \textit{(i.e., W is totally $\mathbb{G}$-free).} \end{aligned}$$

Proof of Theorem B.9. The inclusion $i_W^* F_{\mathbb{G}} \subset F_{\mathbb{G}(W)}$ is trivial (see (5.3)).

Now assume $a \in F_{\mathbb{G}(W)}$. Choose $B = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Sym}^2(T^*)$ where the blocking is induced by the splitting $T \equiv W \oplus N$ with $N = W^\perp$. The set $i_W^* F_{\mathbb{G}}$ is closed (since it is the linear image of a cone with vertex at 0 on a compact set). This makes it possible to assume that $\mathrm{tr}_V a \geq \epsilon > 0$ for all $V \in \mathbb{G}(W)$ (replace a by $a + \frac{\epsilon}{n} I$). Consider the following open neighborhood of $\mathbb{G}(W)$ in $\mathbb{G}$

$$\mathcal{N} \;\equiv \left\{V \in \mathbb{G} : \mathrm{tr}_V B \;>\; \tfrac{\epsilon}{2}\right\}. \tag{B.5}$$

Next we use the fact that for all $V \in G(p,T)$

$$\langle P_V, P_N\rangle \;\geq\; 0 \quad \text{with equality} \qquad \Longleftrightarrow \qquad V \subset W. \tag{B.6}$$

In particular,

$$\inf_{V\in\mathbb{G}-\mathcal{N}} \langle P_V, P_N\rangle \ \equiv\ \delta\ >\ 0. \tag{B.7}$$

Set

$$\inf_{V\in\mathbb{G}-\mathcal{N}} \langle P_V, B\rangle \ =\ -M. \tag{B.8}$$

Then

$$\mathrm{tr}_V(B+tP_N)\ \geq\ -M+t\delta \qquad \text{for}\ \ V\in\mathbb{G}-\mathcal{N} \tag{B.9}$$

while

$$\mathrm{tr}_V(B+tP_N)\ \geq\ \mathrm{tr}_V B\ >\ \tfrac{\epsilon}{2} \qquad \text{for}\ \ V\in\mathcal{N}. \tag{B.10}$$

Thus if $t>>0$ so that $-M+t\delta>0$, then $A\equiv B+tP_N\in F_{\mathbb{G}}$, and of course $i_W^*A = i_W^*B = a$. ∎

Remark B.12. If $F\subset \mathrm{Sym}^2(T^*)$ is a closed convex cone with vertex at the origin (not necessarily geometrically defined), then for each subspace $W\subset T$

$$F+\ker i_W^*\ =\ \mathrm{Sym}^2(T^*) \qquad \Longleftrightarrow \qquad F^0\cap \mathrm{Sym}^2(W^*)\ =\ \{0\}, \tag{B.11}$$

since the polar of an intersection is the sum of the polars, and $\ker i_W^*$ and $\mathrm{Sym}^2(W^*)$ are polars of each other. Thus

$$F^0\cap \mathrm{Sym}^2(W^*)\ =\ \{0\} \qquad \Longleftrightarrow \qquad W\ \ \text{is}\ F\ \text{free}. \tag{B.12}$$

This is useful in the convex cone cases which are not geometric. In the geometric case ${F_{\mathbb{G}}}^0 = \mathrm{ConvexCone}(\mathbb{G})$ is the convex cone on $\mathbb{G}$ with vertex at the origin. This proves that W being $\mathbb{G}$-free can be characterized by either of the following:

$$\mathbb{G}\cap \mathrm{Sym}^2(W^*)\ =\ \emptyset \qquad \Longleftrightarrow \qquad \mathrm{ConvexCone}(\mathbb{G})\cap \mathrm{Sym}^2(W)\ =\ \{0\}. \tag{B.13}$$

Appendix C. Extension Results.

Thus far we have not discussed the extension question:

Given a subequation F on Z and a submanifold $i : X \subset Z$, which i^*F-subharmonic functions on X are (locally) the restrictions of F-subharmonic functions on Z?

The extreme form of this question arises when $i^*F = J^2(X)$, and so every function is i^*F-subharmonic. We address this question in two geometrically interesting cases.

Suppose $F \subset J^2(Z)$ is a subequation each fibre of which is a cone with vertex at the origin (F has the *cone property*). Recall the embedding $\mathrm{Sym}^2(T^*Z) \subset J^2(Z)$ as the 2-jets of functions with critical value zero, and set $F_0 \equiv F \cap \mathrm{Sym}^2(T^*Z)$. In Appendix B we have defined what it means for a subspace $W \subset T_zZ$ to be totally F_0-free (see Definition B.5).

Definition C.1. A submanifold $X \subset Z$ is said to be **totally F-free** if each tangent space T_xX is totally F_0-free.

Remark C.2. In the geometric case considered in Section 8, a submanifold is $F_{\mathbb{G}}$-free if it has no tangent $\mathbb{G}$ planes.

In Theorems C.3 and C.6 we assume that F satisfies the mild regularity condition $\mathrm{Int}(F_x)_0 \subset \mathrm{Int}F$ for each $x \in X$.

THEOREM C.3. *Suppose F is a subequation on Z with the cone property and that $X \subset Z$ is a closed, totally F-free submanifold. Then every $u \in C^2(X)$ is the restriction of a strictly F-subharmonic function $\widetilde{u}$ on a neighborhood of X in Z.*

Now consider a geometric subequation $F_{\mathbb{G}}$ on a riemannian n-manifold Z determined by $\mathbb{G} \subset G(p, TZ)$ as in §§7 and 8.

Definition C.4. A submanifold $X \subset Z$ is strictly $\mathbb{G}$-convex if at each point $x \in X$ there is a unit normal vector n and $\kappa > 0$ such that

$$\mathrm{tr}_\xi \{\langle B, n\rangle\} \ \geq\ \kappa \quad \text{for all } \xi \in \mathbb{G}(T_xX) \tag{C.1}$$

where B is the second fundamental form of X (cf. §8). (This holds if $\mathbb{G}(T_xX) = \emptyset$.)

THEOREM C.5. *Suppose $X \subset Z$ is a strictly $\mathbb{G}$-convex submanifold. Then every $u \in C^2(X)$ is locally the restriction of a strictly $\mathbb{G}$-plurisubharmonic function on Z.*

The proof of Theorem C.3 is based on the following result which has other interesting applications.

THEOREM C.6. *Suppose that X is a closed submanifold of Z, and that $v \in C^2(Z)$ satisfies*

$$X \ =\ \{v = 0\}, \qquad v\ \geq\ 0, \qquad \text{and} \qquad \mathrm{rank}\,\mathrm{Hess}_x\, v = \mathrm{codim}X,\ \forall\, x \in X$$

Then X is totally F-free if and only if the function v is strictly F-subharmonic at each point of X (and hence in a neighborhood of X).

Proof. Fix $x \in X$ and set $B \equiv \mathrm{Hess}_x v$. Then we have

$$B \ \geq\ 0, \qquad B\big|_{T_xX} \ =\ 0, \qquad \text{and} \qquad \text{rank}\, B \ =\ \text{codim}X.$$

If X is totally free, then by Property (3) in Theorem B.2 we have $B \in \text{Int}(F_x)_0$. Now since by assumption we have $\text{Int}(F_x)_0 \subset \text{Int}F$, we conclude that v is strictly F-subharmonic at x. Conversely, v is strictly F-subharmonic at x, then $B \in \text{Int}F_x \cap \text{Sym}^2(T_x^*Z)$ and $B\big|_{T_xX} = 0$. Thus condition $(3)'$ of Theorem B.2 is satisfied, proving that T_xX is $(F_x)_0$-free. ■

Proof of Theorem C.3. Pick any C^2-extension of u to Z and also denote it by u. Let v be a function on Z with the properties assumed in Theorem C.6. We may write $v = \rho^2$ by taking $\rho(z) = \text{dist}(z, X)$ near X for some riemannian metric on Z. Let $\beta : Z \to \mathbf{R}$ be a smooth extension of a given positive function on X, and set $\widetilde{u} \equiv u + \beta\rho^2$. Then we compute that *along the submanifold* X:

$$d\widetilde{u} \ =\ du \qquad \text{and} \qquad D^2\widetilde{u} \ =\ D^2u + \beta D^2(\rho^2).$$

That is, along the submanifold X:

$$J(\widetilde{u}) = J(u) + \beta J(\rho^2).$$

At each point $x \in X$ we have $J_x(\rho^2) \in \text{Int}(F_x)_0 \subset \text{Int}F$. Therefore by choosing the positive function β to be sufficiently large at each point $x \in X$, we will have $J(\widetilde{u}) \in \text{Int}F$ along X, and therefore on a neighborhood of X in Z. ■

Proof of Theorem C.5. Fix $x \in X$. It is straightforward to see that by strict $\mathbb{G}$-convexity, there is a smooth unit normal vector field n defined in a compact neighborhood V of x on X and a $\kappa > 0$ so that (C.1) holds at all points of V.

For simplicity we rename V to be X. For clarity we restrict to the case where Z is euclidean space $\mathbf{R}^n$ Consider the tubular neighborhood

$$U \ \equiv\ \{x + \nu \in \mathbf{R}^n : x \in X, \nu \in B_\epsilon(0), \nu \perp T_xX\}$$

for some small $\epsilon > 0$, and define a function f on U by

$$f(x+\nu) \ =\ \langle n(x), \nu\rangle + \tfrac{1}{2}c|\nu|^2$$

where $c > 0$ will be determined later. Set $\rho(x+\nu) \ =\ \langle n(x), \nu\rangle$. Note that $\rho \equiv 0$ on X and therefore

$$\text{Hess}_X\rho \ \equiv\ 0.$$

From formula (8.1) we see that

$$\text{Hess}_{\mathbf{R}^n}\rho\big|_{TX} \ =\ \langle B, n\rangle \quad \text{on} \quad X. \tag{C.2}$$

One easily sees that the Hessian of $\frac{1}{2}|\nu|^2 = \text{dist}(\bullet, X)^2$ is

$$\tfrac{1}{2}\text{Hess}_{\mathbf{R}^n}|\nu|^2 \ =\ P_N \ \equiv\ \text{orthogonal projection onto the normal space to } X \tag{C.3}$$

It follows that

$$\mathrm{Hess}_{\mathbf{R}^n} f\big|_{TX} \;=\; \langle B, n\rangle.$$

Hence, by (C.1) we have

$$\mathrm{tr}_\xi \left\{\mathrm{Hess}_{\mathbf{R}^n} f\right\} \;\geq\; p\kappa \;\;\text{ for all } \xi \in \mathbb{G}(TX),$$

and therefore there exists a neighborhood $\mathcal{N}$ of $\mathbb{G}(TX) \subset \mathbb{G}\big|_X$ so that

$$\mathrm{tr}_\xi \left\{\mathrm{Hess}_{\mathbf{R}^n} f\right\} \;\geq\; \kappa/2 \;\;\text{ for all } \xi \in \mathcal{N}.$$

Now for a general $\xi \in \mathbb{G}\big|_X$,

$$\mathrm{tr}_\xi \left\{\mathrm{Hess}_{\mathbf{R}^n} f\right\} \;=\; \mathrm{tr}_\xi \left\{\mathrm{Hess}_{\mathbf{R}^n} \rho\right\} + c\langle P_\xi, P_N\rangle$$

and by compactness there exists $a > 0$ so that

$$\langle P_\xi, P_N\rangle \;\geq\; a \;\;\text{ for all } \xi \in \mathbb{G}\big|_X - \mathcal{N}.$$

Let

$$b \;=\; \inf_{\xi\in\mathbb{G}\big|_X} \mathrm{tr}_\xi \left\{\mathrm{Hess}_{\mathbf{R}^n} \rho\right\}.$$

Then for $c > 2|b|/a$ we have

$$\mathrm{tr}_\xi \left\{\mathrm{Hess}_{\mathbf{R}^n} f\right\} \;>\; |b| \;\;\text{ for all } \xi \in \mathbb{G}\big|_X.$$

It follows that

$$\mathrm{tr}_\xi \left\{\mathrm{Hess}_{\mathbf{R}^n} f\right\} \;>\; |b| \;\;\text{ for all } \xi \in \mathbb{G}\big|_W$$

where W is a neighborhood of X.

Now suppose we are given $u \in C^2(X)$ and $x \in X$. Pick any C^2-extension of u to a neighborhood of X and denote it also by u. On a small compact neighborhood V of x in X apply the construction above to produce the function f on a neighborhood of V. Then for λ sufficiently large, the function $\widetilde{u} \equiv u + \lambda f$ will be strictly $\mathbb{G}$-psh on a neighborhood of V and satisfy $\widetilde{u}\big|_V = u$.

For the case of a general riemannian manifold Z, we use the exponential map to identify the normal bundle of X with a tubular neighborhood of X in Z, and to the analogous construction. ■

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